USE OF A NEUTRALIZATION MODEL TO DEVELOP UNDERSTANDING OF INTEGERS AND OF THE OPERATIONS OF INTEGER ADDITION AND SUBTRACTION

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in partial fulfillment of the requirements
for the degree of Master in the Teaching of Mathematics
Concordia University
Montreal, Quebec, Canada

September 1992

ISBN 0-315-80946-9

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Patricia Ann Lytle

Entitled: Design and Experimentation of a Teaching Outline Based on a Neutralization Model for the Construction of the Notions of Integers, Integer Addition and Subtraction.

and submitted in partial fulfilment of the requirements for the degree of

Master in the Teaching of Mathematics

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

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ABSTRACT

USE OF A NEUTRALIZATION MODEL TO DEVELOP UNDERSTANDING OF INTEGERS AND OF THE OPERATIONS OF INTEGER ADDITION AND SUBTRACTION

Patricia A. Lytle

Assessment studies from around the world have indicated student difficulties with integer operations, in particular with that of integer subtraction. The most commonly used model for integers and their operations, the number line, often in combination with the traditional subtraction rule, has been inadequate to produce understanding and performance. An alternative model, one based on neutralization of positive and negative quantities, was investigated in this study, and an individualized teaching experiment using this model was conducted with four grade seven students. Results of this teaching experiment are encouraging with respect to the construction of understanding, but not all obstacles were overcome. This study may be viewed as a pilot study which may be refined for further use.

ACKNOWLEDGEMENTS

I deeply appreciate the assistance that the professors in the MTM program with whom I've had the privilege to study have given me. They have each made unique contributions to the development of my awareness of research issues, as well as provided opportunities for me to be involved in various research projects during the time of my studies. In particular, Professors Nicolas Herscovics and Anna Sierpinska, the co-directors of this thesis, deserve special thanks. I am grateful for the guidance, wisdom and attention to detail of Dr. Herscovics, who despite poor health gave graciously of his time when he was able. I am indebted to Dr. Sierpinska for her guidance in the theoretical framework and her assistance with the teaching experiment.

I also wish to express my appreciation to my fellow students, who in an atmosphere of supportive co-operation shared ideas, encouragement and constructive criticism. In particular, I am indebted to Louis Stevens for allowing me access to his Secondary I class at Pierrefonds Comprehensive High School, and for making all of the necessary arrangements at the school. His students Steven, Robert, Connie and Jason, with their parents' consent, kindly agreed to be subjects in this study.

I acknowledge the generosity of Prof. Herscovics and the Quebec Ministry of Education in funding my research with an FCAR grant (#92ER1032).

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INTRODUCTION

This research investigation is concerned with two separate, yet complementary themes: the ability to correctly perform integer operations, and the construction of understanding of integer notions. As early as the seventh century AD rules existed in India for integer calculations (Boyer 1968, Smith 1925), yet even within the mathematical community, many obstacles existed which prevented the acceptance of negative numbers as numbers in their own right. It is generally agreed that it was not until Hankel in 1867 that full understanding of negative numbers, and integers as a set of numbers, was complete (Glaeser, 1981).

Recent assessment studies from around the world have alerted educators to present-day student difficulties with operations on integers, especially with integer subtraction. Chapter One of this thesis shows some of the findings of these studies, which serve as a motivation to attempt to find a way to help students learn about integers and the operations of integer addition and subtraction. Chapter Two explains the methodology used to investigate this problem.

A description of the additive set of integers forms a basis for criteria of understanding of integers as a number system. This description, along with an investigation of obstacles to the learning of integers, is given in Chapter Three.

Chapter Four examines the use of models to teach mathematical concepts, and gives a detailed description of the two models currently used by most teachers and textbooks to teach integer concepts. It is shown that the model most commonly, in fact almost exclusively, used, the number line, has been brought into question as a teaching tool, and seems to be inadequate especially for integers. The alternative model, one based on neutralization of opposites and on equivalent representations of integers, seems to hold promise as a tool for teaching for understanding of integers and the operations of integer addition and subtraction.

This thesis investigates whether the neutralization model, described in Chapter Four, indeed promotes understanding of integers in students, and improves pupil performance especially on integer subtraction. The method of investigation used combines an a priori analysis of the concept of integer with a teaching experiment. Chapter Five contains the epistemological analysis of integers and of integer addition and subtraction and its implications for the planned teaching experiment involved in this research.

In order to determine which integer notions are already understood by students in grade 7 before formal instruction, and which skills have been attained, a pre-test was designed and administered to 3 classes of grade 7 students, as well as a class of grade 6 students. Following this, six of the 12 year olds from grade 7 were interviewed in order to probe more deeply into their understanding of integers. The design and results of the pre-test and interviews are found in Chapter Six.

A teaching experiment based on the a priori analysis was designed to teach integer notions and the operations of addition and subtraction using the neutralization model as the main tool, and this experiment was tested on four individual students with no previous formal instruction about integers. The design of the experiment, as well as the rationale for the design, is found in Chapter Seven, along with the assessment of the teaching experiment with respect to understanding of integers and their operations using the neutralization model.

Chapter Eight contains conclusions and recommendations regarding the use of the neutralization model to promote understanding of integers and their operations.

CHAPTER ONE

DIFFICULTIES WITH INTEGERS

1.0 INTRODUCTION

A survey of current findings concerning how students perform on addition and subtraction tasks involving integers can reveal, by success rates on particular tasks, the types of tasks which are more difficult for students than others. These surveys do not measure understanding of integer concepts, such as the notion of oppositeness. This latter type of investigation, which is more difficult to conduct in a written format, and with a large sample of students, appears less frequently in the literature.

1.1 CHILDREN'S PERFORMANCE WITH INTEGER OPERATIONS

1.1.1 Assessment Studies

Studies assessing performance of large numbers of students on various mathematical tasks have been conducted in many countries. Table 1 outlines relevant information about some of these studies, and provides references.

Table 1: Large-Scale Mathematics Assessments

STUDY	YEAR(S)	LOCATION	SAMPLE SIZE	REFERENCE
NAEP 1	1972-73	USA		Carpenter et al., (1978)
NAEP 2	1977-78	USA	2400 each of age 13, 14, 15	Carpenter et al., (1981)
NAEP 3	1982-83	USA		unreported
NAEP 4	1985-86	USA		Lindquist (1989)
CSMS	1974-79	Britain	818 aged 13, 14, 15	Bell et al. (1983) Kuchemann (1980,1981)
APU	1980?	Britain	15 year olds?	Bell et al. (1983)
IEA 1	1964	12 countries		
IEA 2	1982	24 countries	1652 gr. 8	Chang & Ruzicka (1985) Crosswhite et al. (1986)
S. A. - 1	1985?	Cape Town	993 grade 8, 1331 gr. 9	Murray (1985)
S. A. - 2	1985?	country town	514 grade 9, 332 gr. 10	Murray (1985)
S. A. - 3	1987	Kimberley	97 grade 7	Human & Murray (1987)

Legend: NAEP = National Assessment of Educational Progress
CSMS = Concepts in Secondary Mathematics and Science Study
APU = Assessment of Performance Unit (secondary students)
IEA = International Association for the Evaluation of Educational Achievement
S. A. = South Africa

Although several of these assessments tested very few integer tasks, these studies have revealed a consistent trend of lower success rate (by about 20%) on integer subtraction items than on integer addition items on the same test. It is difficult to compare among studies, since some tested students who had received little (grade 8 S. A.) or no (grade 7 S. A.) formal instruction about integer operations. Others tested a mixture of students with and without instruction (a number of 13 year olds in Britain and the United States), while other studies used groups which had received full instruction in integer operations (the group tested by Lytle and Avraam who were all from grades 8, 9 and 10). Despite the fact that some test items, such as those in the CSMS study which actually <u>taught</u> the use of the number line for addition and the procedure of adding the additive inverse for subtraction (Kuchemann, 1975), can be challenged due to the format of the testing (Lytle & Avraam, 1990), it is clear that many students have difficulty with operations on integers, and in particular with integer subtraction. It should be noted here that some of the surveys also tested multiplication and division, but these operations are beyond the scope of this thesis.

Harvey and Cunningham (1980) report results of an integer assessment test given to 163 grade 8 Montreal students, using a total of 46 questions covering most possible combinations of addition and subtraction, as well as both one digit and two digit integers. In addition, 62 female students of ages 13, 14 and 15 in the high school where the author had been teaching were given 6 integer addition and 6 integer subtraction items, covering several possible combinations and outcomes for both operations (Lytle &

Avraam, 1990). Results from these two smaller assessments were comparable to those of the large scale assessments in that performance on integer subtraction items was considerably lower than that on addition items.

The only evidence of testing using a "neutralization" or "colored number" model is Janvier's (1983,1985) account of a survey of Moroccan students. Although the exact test items are not given, we see that this is the only situation where results on subtraction items were actually better in about half of the cases than results on addition items. We are told that in this study subtraction was taught as the "addition of opposites" rule, yet it is not evident that this procedure was followed, as for example the easier subtraction $^{+}L - {}^{+}S$ (88%) was better answered than the corresponding addition $^{+}L + {}^{-}S$ (80%), and the subtraction $^{+}L - {}^{-}S$ (73%) was more poorly answered than the easier addition $^{+}L + {}^{+}S$ (88%). Note that here, and throughout, the capital letters L and S refer to larger and smaller respectively, ordering determined by absolute value. Thus $^{+}8 - {}^{-}2$ is an example of $^{+}L - {}^{-}S$.

A summary of results on one-digit items from all studies mentioned is found in Appendix A, and illustrates the various rates of success for both addition and subtraction of integers. It should be noted that only isolated items were selected for this analysis, since many tasks were number line bound, and Ernest (1985) has suggested that this type of item tests the ability to use the number line, rather than the ability to perform operations on numbers. It is interesting to look more closely at the two items from the CSMS test of addition of two positive integers. The task $^{+}2 + {}^{+}6$ received 95% to 97% success, while the similar task $^{+}2 + {}^{+}4$ received only 60% to 64% success. In examining the test itself (Kuchemann, 1975), it can be seen that the first item is grouped with exclusively addition items, immediately following structured teaching within the test of addition on a number line. The second item, however, is found at the end of the test, grouped with other items representing all four operations.

In general, the only addition items which received a success of less than 60% were those of additions of integers of different signs, and this was for the grade 7 and 8 groups tested in South Africa, but the first group had had no integer instruction, and the second group of students had only received a 20 minute introduction to negative numbers through reference to the thermometer before being assessed. Many subtraction items scored well below a 50% success rate, in particular those which involved differences of integers of unlike signs.

1.1.2 Interviews With School Children

Seven of the students from the author's school (4 successful, 3 unsuccessful with integer subtraction items, but all successful with integer addition) were interviewed in an effort to reveal the cause of difficulty with the subtraction items. It was found that the students either used, or were able to confirm their addition strategies with, a physical model, in particular the number line. However, for subtraction items, those who were successful relied on the rule "add the opposite" or "change both signs", but did not understand why the rule worked, and were not able to refer to a physical model to validate their results. Those who had been unsuccessful at subtraction were unable to remember the rule, and had no physical model to fall back on (Lytle & Avraam, 1990).

Bell (1982) also interviewed 25 students aged 15 and found that 20 were successful on all addition tasks, mainly by referring to a number line or to quantity. However, only 10 were successful on all subtraction items, success depending on correct or incorrect application of learned rules.

These two studies suggest that for these students, addition has a semantic reference, but subtraction is only referenced by a rule.

1.2 CHILDRENS' UNDERSTANDING OF NEGATIVE NUMBERS

In order for students to develop meaning for integer operations, there must be some intuitive feeling for negative numbers (Bell, 1983). There has been little investigation into this understanding, but the following has been gathered from anecdotal events and from interviews with school children.

(a) Children who have received little or no formal integer instruction.

Some authors (Brown 1986, Cochran 1966, Mertz 1979) provide anecdotes about individual members of their class or family who at ages between 4 and 7 display a knowledge of the existence of positions on the other side of zero. These positions are symmetrical to those of the counting numbers, either on a real number line or a mental one. These children do not know the notation or terminology (using "seven less", "2 less than 0", "a backward one"), but adopt it when given the correct words or notation, and can perform operations which may be translated into moves along this extended number line (a score going down by 2 for example). One child in grade 1 (Cochran, 1966), when playing a number game between teacher and class, offered a negative number as an input value (a day after the notion of numbers beyond zero had arisen in class), suggesting that negative numbers had the validity of being "number" for the child.

Aze (1989) reports an incident with a class of 7 year olds who were acting out movements (e.g. 3 steps forward) on a playground number line, and who were able to invent new positions beyond the zero, assigning non-numerical labels (names of students) to these new positions.

Murray (1985) interviewed 52 students aged 9 to 13 after a brief temperature-related introduction to integers. He found that "the concept of negative number is probably not alien to the experience of many young students". Some meanings given for the notation $^-5$ were "5 below zero", "less than zero", "you have to get 5 before you have zero". They performed computations by "extrapolating from known facts about positive number arithmetic", some using a number line (but starting with the negative number); others reasoning about tasks like $^-5 - {^-2} = {^-3}$ as normal subtractions like $5 - 2 = 3$; and some using opposite operations since negative numbers are opposite to positive numbers.

Peled, Mukhopadhyay & Resnick (1989) interviewed 6 female children from each of grades 1, 3, 5 (who had no previous instruction), 7 and 9 (after instruction in grades 6 and 7) following a test written by all students in those grades, in order to investigate what informal understanding these students had about negative numbers. The first graders did not show evidence of the existence of these numbers, as they always responded with a positive answer to operational tasks, and ignored the negative sign attached to any number. However, students in grades 3 and 5 were more likely to generate a negative number as a solution to 5 - 7 (which the authors feel showed that these students believed in the existence of negative numbers), and up to half of these students were able to solve many of the operational tasks. Almost all of the fifth graders correctly ordered $^{-}4$ and $^{-}6$ ($^{-}4$ is greater than $^{-}6$), which the authors claim infers the construction of a mental extended number line, symmetric about zero. Peled et al. conclude that their study "provides clear evidence that many children construct mental models that include negative numbers before school instruction on this topic is offered".

(b) Children who have received formal integer instruction

Peled (1991) interviewed 20 grade 6 students (10 strong, 10 weak) who had received integer instruction the previous year, although many had forgotten much of the instruction. Some of the notions which emerged were the following:

- since one gets to a negative number by going left, then adding a negative number should also mean going left (identifying negative with left)

- a large natural number cannot be subtracted from a smaller one, since "nothing will come of it"

1.3 IMPLICATIONS OF FINDINGS

Poor performance on integer subtraction tasks appears to be the result of the absence of a concrete model which represents subtraction in a meaningful way, and which can be accessed as a mental model for large number tasks. Even though younger children appear to be able to broaden their natural number concepts to include negative numbers in an extended mental number line, this model seems to be meaningful for addition only, and the use of the traditional subtraction rule does not seem to be adequate to produce either understanding of or success with this operation.

These findings motivate research into the proposal of an alternative concrete model for representing integers which could help students construct meaning for integers and the operations of integer addition and subtraction, and could act as a mental model for integer tasks of these types.

CHAPTER TWO

METHODOLOGY

2.0 RESEARCH PROBLEMS

Poor results on subtraction items in large-scale assessments and lack of documented evidence of studies on students' understanding of integers led to the following research problems for investigation:

[1] What does it mean to understand integers and integer addition and subtraction? What obstacles have contributed to a lack of understanding?
[2] How is the current model for integer instruction (the number line) used, and where are its inadequacies (especially for subtraction)?
[3] What are the alternate models which could be used to construct meaning for integers and their operations? Which model should be pursued for this purpose?
[4] How is the chosen alternate model used, and where are its strengths and weaknesses?
[5] What does it mean to understand integers within the context of this alternate model?
[6] What meanings have grade 7 children already constructed for negative numbers and their operations? How can this be assessed?

Based on the findings of answers to these problems, it remains to design a teaching outline to address the implementation of the alternate model to the construction of understanding of integers and the operations of integer addition and subtraction. Following this, the teaching design must be experimented with students, and the results of the teaching experiment analyzed.

Methodology for finding solutions to the above problems is outlined in sections 2.1 through 2.7.

2.1 EPISTEMOLOGICAL ANALYSIS OF INTEGER UNDERSTANDING

The additive group of integers (Z,+) has certain properties when considered as a formal abstract structure, and it was desired that these properties should form a basis for the less formal study of integers as a number system, so that at a later stage of instruction the formal notions could be readily accessible to the student from the initial "informal" understanding. A brief description of the properties of Z was used to develop criteria for understanding of integer numbers. These criteria would later serve as a standard by which to measure integer "content" of two instructional models.

A review of the literature was conducted in order to reveal some of the obstacles to the understanding of integers which have been documented not only for school children, but also in the historical development of the set of integers. The design of the class assessment and interview would be done in such a way as to include items which might reveal the presence of the obstacles appropriate to integers as numbers. In addition, the teaching experiment would confront the students with these obstacles in order to assist in overcoming them.

2.2 INTEGER MODELS

In view of the difficulties with integer operations which were observed in Chapter One of this thesis, it was decided to review the literature in order to conduct a detailed investigation of the number line model's approach to integers. An analysis of how the model is used for addition and subtraction in both W (whole numbers) and Z was made in order to uncover obstacles within the model that could be responsible for some of the difficulties. Authors' opinions, both positive and negative, of the use of the number line were also sought and documented in order to expose the advantages and disadvantages of its use.

A literature review of an alternative model, one based on the concept of neutralization of opposites, was deemed necessary in order to fully examine all features of this second model. It was thought that a comparison of various authors' interpretations of the key concepts of the model (opposites, neutralization, etc.) could aid in developing a comprehensive version that could be used as an improved version in a teaching experiment, with all components fully and clearly developed. This composite version could then be analyzed regarding the nature of its presentation of integer concepts and procedures for addition and subtraction. Again, authors' opinions of the use of the model were thought to be of benefit in order to determine the value of the model.

Both the number line and the neutralization models were compared using the criteria established for understanding of integers as a number system in order to establish whether one model was preferable over the other in this regard. The presence of essential features of Z as well as the existence of irrelevant features were noted.

A number of current grade 7 textbooks were examined in order to investigate whether any of them proposed the use of the neutralization model, and to observe the extent of their use of the model. In addition, four curriculum documents were also reviewed in this manner. This investigation was conducted in order to ascertain how the model has been interpreted by these educators, and to evaluate whether or not this model in the form presented in these documents would be adequate to teach all of the concepts essential to and extending from the model.

The literature was also surveyed in order to uncover any previous research conducted on the use of any version of the neutralization model with students. This was done in order to determine what effects, both positive and negative, the model has shown to have had on the learning of integers, and to expose areas for further research that the author could undertake.

2.3 UNDERSTANDING OF INTEGERS USING THE NEUTRALIZATION MODEL

A literature review was conducted in order to ascertain what criteria authors had used to assess student's understanding of integers. Only one type of assessment was found, one that used performance-based levels of understanding, with reference to either the number line or to a representation of integers as good and bad quantities.

The author of this current research endeavored to propose an analysis of the understanding of integers and the operations of integer addition and subtraction with reference to the neutralization model, using the criteria for understanding established earlier. Items resulting from the analysis were structured into levels of understanding based on a two-tiered model of understanding created by Herscovics & Bergeron (1988). Implications for assessment of student understanding as well as for the teaching of integers and the operations of integer addition and subtraction were made from this analysis in preparation for an assessment study and for a teaching experiment. This structure would also be used to determine the students' level of understanding achieved as a result of the instruction.

2.4 ASSESSMENT OF INTEGER SKILLS AND UNDERSTANDING

A written assessment (pre-test) was designed in order to evaluate the kind of informal knowledge some grade 7 students had constructed (correctly or incorrectly) from any previous exposure to negative numbers. Questions were designed to assess familiarity with notation for negative numbers, informal meaning of a negative number, and addition and subtraction performance (on items containing a negative number and a whole number). In order to assess readiness for the neutralization model, questions were also designed to assess familiarity with the terminology associated with neutralization, and also to assess mathematical procedures on whole numbers necessary for mathematical processes within the model. This assessment was administered to approximately 100 children from four classrooms by their teachers.

Due to the difficulty of assessing thinking processes from paper-and-pencil tasks, semi-standardized interviews were conducted with six students who had written the pre-test. These students were chosen from 50 available for further study by means of teacher-classified ability, responses to particular tasks on the written assessment, and an arbitrarily chosen age (12). Interview tasks were similar to some on the pre-test, but were approached in a different manner. The semi-standardized style of interview was chosen since it permits one to pursue different thinking processes of individual children while maintaining a uniform script for all tasks presented. The interviews were tape-recorded and transcribed, and analysis was conducted from the transcripts as well as from the students' written work. From the six students who were interviewed, four were chosen for the teaching experiment.

2.5 TEACHING OUTLINE

A series of five 40-minute lessons was designed, with content for each developed from the analysis of integer understanding within the context of the neutralization model. Concepts were grouped into the following five units, each unit comprising one lesson: introductory notions, addition, subtraction, ordering, generalizations. Each lesson was designed to present the concepts within the framework of the concrete model, then to lead to the mathematical representation of the concept. Although it was felt that one lesson might not provide enough exposure to certain topics (subtraction, generalizations), the students involved in the study were being excused from regular classes to participate in the study, and there was a concern that interruptions in their classroom instruction be kept to a minimum.

Each lesson after the first began with a review of earlier notions in order to assess acquired skills and understanding, and to allow for an opportunity for remediation if necessary. A sixth meeting with the students took the form of a post-test, in which questions were designed to assess the generalizations made about some of the ideas from the previous sessions, as well as performance on integer tasks.

2.6 TEACHING EXPERIMENT

An individualized format as opposed to group or class instruction was chosen so that the thinking processes of each student could be revealed as the instruction was taking place.

Another advantage of this format is that it may serve to uncover specific difficulties with the model of instruction. Each semi-standardized lesson was audiotaped and transcribed so that analysis could be made from the childs' verbal responses as well as from written work.

The lessons were each analyzed from the point of view of the ease of use of the concrete model, the transition from the model to the related integer concept, the students' ability to develop skills using the model, and the level of the students' understanding of the integer concepts.

2.7 CONCLUSIONS AND RECOMMENDATIONS

The neutralization model was assessed with respect to the results of the teaching experiment regarding the research issues of the construction of the understanding of integers as well as the operations of integer addition and subtraction. Other related issues, such as the introduction of the subtraction rule (i.e. "add the opposite"), were also evaluated as to their significance in light of the neutralization model.

CHAPTER THREE

EPISTEMOLOGICAL ANALYSIS OF THE UNDERSTANDING OF INTEGERS AS A NUMBER SYSTEM

3.0 EPISTEMOLOGICAL ANALYSIS OF INTEGER UNDERSTANDING

An investigation of what it means to understand integers must first deal with the question: What are integers? What are these mathematical objects that we want students to understand? In answer to this question, two distinct concepts emerge - that of the formal algebraic structure, and that of an informal number system. Integers have an existence in mathematics, as an abstract structure, the additive group (Z,+). On the other hand, in everyday life, one speaks of negative numbers. Positive numbers and negative numbers are not conceived of as a structure, but negatives are thought to be "negative counterparts" of whole numbers, useful for describing certain situations like temperatures, debts, altitudes below sea level, etc. One can call this latter view of integers a "concrete approach", since it is a use of specific isolated negative numbers. These negative numbers are not in reality an extension of the natural numbers, but along with zero and the positive numbers, form a set _isomorphic_ to the natural numbers, where Z^+ is not equal to N.

From the point of view of abstract algebra, integers are a set of numbers, **ordered linearly**, and **closed** under addition, subtraction and multiplication, but not division. Within this abstract view of integers, the notions of **identity element for addition** (i.e. 0: $n + 0 = 0 + n = n$), **inverse elements for addition** (i.e. the opposites n and ^-n: $n + {}^-n = {}^-n + n = 0$) and **equivalence classes** ($(m,n) = (m+y,n+y)$ where m, n and y are natural numbers) are essential features. Addition is defined as follows: $(a,b) + (c,d) = (a+c,b+d)$, and ordering may be expressed as $(a,b) > (c,d)$ if $a+d > b+c$. Any integer n may be either positive or negative, and thus ^-n does not necessarily represent a _negative_ number, but in fact the _opposite_ of n (Hall, 1974). Note that this last statement is one which causes conflict to the algebra student who does not expect ^-n to be positive. Order is established as $b > c$ if there exists a natural number m such that $b = c + m$, where b and c are integers.

From the concrete viewpoint, integers are numbers such as $^{-}2$, $^{+}4$, $^{-}100$. These numbers are *signed* numbers, either positive or negative, with the exception of zero, which is neither positive nor negative, and has no sign. Integers are numbers which *quantify* positive and negative situations using zero as a reference point, such as degrees above or below zero, altitude above or below sea level, etc. Negative integers may emerge as solutions to equations such as $2 - 8 = \square$, but they are just positions on an extension of the natural number line. Integers are *opposites*, and each integer is the opposite of another integer which has the same magnitude but the opposite sign (the opposite of $^{-}5$ is $^{+}5$ and the opposite of $^{+}2$ is $^{-}2$). These opposites are **inverse elements for addition**, so for example, $^{+}3 + {}^{-}3 = {}^{-}3 + {}^{+}3 = 0$. The exception to this is zero, which has no opposite, and is therefore "neutral", and is in fact is the **identity element for addition** (for example $^{-}6 + 0 = 0 + {}^{-}6 = {}^{-}6$). **Order** is established by position on a number line (the number to the right is larger than the one on the left). In this set of numbers, all subtractions are possible (**closure**). Any particular integer represents the **equivalence class** of all ordered pairs whose canonical representation is the same as that of the integer (e.g. $^{+}3 = (3,0) = (4,1) = (18,15) = (103,100)$, etc.).

One must now consider the 12 year old student who is just entering the Piagetian Formal Operational stage, where he is just beginning to understand abstract concepts. McAuley (1990) reminds us that "In the classroom development of directed numbers, a mathematically sound development need not be so abstract as to be beyond the majority of pupils". Within this framework the abstract notions of integers will be abandoned in favor of those of the concrete integer numbers.

3.1 UNDERSTANDING OF THE CONCRETE CONCEPT OF INTEGERS

Regardless of the model used to represent these numbers, the following notions about integers are fundamental.

[1] Understanding of Zero

(a) Identification of zero as origin or boundary, and not as an "absolute" zero, as it is in the set of whole numbers. The zero position may be determined arbitrarily, but from this position, numbers extend symmetrically in both positive and negative directions, to be a certain magnitude (distance) "above" or "below" zero.

(b) Identification of zero as the neutral element, since it is (i) neither positive nor negative (ii) neutral in action.

(c) Identification of zero as "nothing", since it has no magnitude. This is similar to a notion of zero in W, where it is the number that represents "no quantity" (6 - 6 = 0 and 5 + 0 = 5 both have the notion of zero as nothing - nothing left, nothing added).

[2] Understanding of Integer Notation

Regardless of the notation used (i.e. raised sign before numeral as in $^{+}3$; brackets around sign and numeral as in (+3); or sign at level of operations sign before numeral as in +3, there should be an identification of the meaning of this sign when the integer is isolated from any numerical operational context, as an indication that (a) this number is an integer (b) this is the "sign" of the integer, specifying whether the magnitude is positive or negative.

In addition, when the integer is encountered in an operational setting, (e.g. $^{+}2$ - $^{-}4$) there must be a discrimination between the sign indicating the operation, and the sign of the integer.

[3] Understanding of Oppositeness of Integers

Identification of a negative number (e.g. $^{-}7$) as the opposite of a positive number with the same magnitude (e.g. $^{+}7$), and vice versa, and the awareness that the criteria for (or result of) their oppositeness is their combination resulting in zero.

[4] Understanding of the Orders in Integers

Integers are ordered by direction ($^{-}3$ is greater than $^{-}7$ since it is more "positive"). There is also a natural number ordering which may be applied to integers to determine "quantity" (i.e. "larger" or "smaller") where the <u>absolute value</u> determination declares that $^{-}6$ is "smaller" than $^{-}10$, since $6 < 10$.

[5] <u>Understanding of Outcomes of Integer Operations</u>

Regardless of the procedure used to perform any of the operations on integers, the following generalizations may be made:

(a) Addition of Opposites results in zero. (e.g. $^{-}12 + {}^{+}12 = 0$)

(b) Both addition and subtraction will produce either an increase in, decrease in, or will have no effect on the value of an initial integer.

(c) Addition and subtraction are inverse operations in that one will "undo" the other. (e.g. $^{+}3 + {}^{-}4 = {}^{-}1$ implies that $^{-}1 - {}^{-}4 = {}^{+}3$).

(d) There are two ways to start from a given number to obtain the same result (e.g. $^{-}6 + {}^{+}2 = {}^{-}4$ and $^{-}6 - {}^{-}2 = {}^{-}4$).

3.2 OBSTACLES TO THE UNDERSTANDING OF INTEGERS

Some authors (Glaeser 1981, Pycior 1984, Schubring 1986) have investigated in detail the history of the acceptance of negative numbers in different countries and mathematical communities throughout the world. Although rules for operating on negative numbers were used, the numbers themselves were not thought to exist. When a negative number emerged as a solution to an equation such as $100 + x = 50$, the equation was changed to read $100 - x = 50$ to allow for a positive solution. The following have been identified by these authors as historico-epistemological obstacles to the acceptance of negative numbers, predominantly in an algebraic context, for mathematicians.

(i) Isolated negative numbers had no meaning attached to them, and were therefore avoided for some time.

(ii) Zero was seen as an "absolute" zero, and nothing could be below zero.

(iii) There was no conception of a continuous number line in both directions, but rather two separate lines were used next to each other.

(iv) Procedural rules were an abstraction which could not be illustrated in a concrete way.
(v) Number as a unit for measuring magnitude was more of a focus than the concept of the number.

Hefendehl-Hebeker (1991) is concerned that these may also be obstacles for today's students who have not yet developed a concept of negative number. For example, the student who can correctly solve 3 - 7, but gives 6 + $^{-}$10 equal to 6 (because to him the addition of a negative number has the same effect as the addition of zero) may recognize negative positions on the number line (obstacle (iii) overcome), but may regard zero as absolute with respect to quantity (obstacles (ii) and (iv)). All of the above authors advocate an abstract approach to the set of integers, yet this does not seem viable for students have not yet reached the Formal Operational stage. In addition to the historical obstacles which may cause these students difficulties, other obstacles have been documented:

(i) Notation causes confusion between the sign of the operation and the sign of the number when superscripts are not used (McAuley 1990). The vocabulary of "minus" and "negative" if not used consistently for subtraction and for numbers less than zero respectively can also cause confusion (Hall, 1974, Shawyer, 1985).
(ii) The practise of not writing the positive sign for positive numbers creates confusion between the numbers in N and the positive numbers in Z. The problem 5 - 8 is not an integer problem when written with unsigned numbers, and as such has no solution unless the isomorphism with Z^+ is invoked, since 5 and 8 are members of N.
(iii) The notions of natural numbers produce conflict with those of the integers (for example that subtraction makes smaller).
(iv) The application of formal rules (such as the subtraction and multiplication rules) when the student is still at a Concrete Operations level.

3.3 IMPLICATIONS FOR INSTRUCTION

Any instruction given to the grade 7 student about integers must be sensitive to the concrete concepts of integers rather than the abstract concepts, while considering the relationship between the two viewpoints. One must be aware of the obstacles which have emerged in the history of the acceptance of negative numbers, so that the teaching design may confront those issues which might be at stake for these students.

CHAPTER FOUR

MODELS USED FOR THE TEACHING OF INTEGERS

4.0 TYPES OF MODELS USED FOR INTEGER ARITHMETIC

Abstract mathematical ideas are often initially embodied in a concrete (colored bingo chips), mental (debts and assets) or symbolic (ordered pairs) framework in order to enable a child to construct mathematical knowledge by transferring the notions and procedures of the model to the mathematical situation. These models must not be artificial, but must relate to the child's own experiences (Janvier, 1983), and must be a temporary tool used to initiate learning, then be forgotten, yet available to be recalled in the future in order to justify the mathematical abstractions (Janvier, 1985).

Greeno (1991) states that students develop number sense, which he defines as "flexible numerical computation", "numerical estimation" and "quantitative judgement and inference", by constructing a mental model based on the concepts that they know within a physical model, where objects in the mental model (in this case, integers) correspond to objects they have become familiar with. Within this mental model, one is able to mentally perform operations such as combining ("corresponding to addition") and separating ("corresponding to subtraction") with the mental objects in the same way in which they had performed the same operations on the physical objects.

Models which are used to help students to understand integers fall into two main categories - the "directed number" model, and the "integers as discrete opposites which can be neutralized" model. The number line is the most common example of the first kind of model, and a mental number line, such as a thermometer scale or the concept of height above or below sea level is often used as well. This model features zero as an origin, and positive and negative integers as positions arranged symmetrically about zero. Integers can also be viewed as vectors of a certain distance or length. In fact, the number line can be used in a totally vector embodiment for integers with respect to addition and subtraction (Eastwood 1983, Freudenthal 1983, Grignon 1989).

The second kind of model uses either physical objects or abstract objects as embodiments of integers. With physical objects, such as bingo chips in two different colors, integers represent quantity of a positive or negative nature, while with abstract objects, such as ordered pairs, integers represent an equivalence class of some combination of positives and negatives. The operations of addition and subtraction merely combine quantities, or remove a quantity respectively. Thus these two kinds of models are semantically different (Janvier, 1983). Models which combine elements of both of the above are the models of debts and credits (Mukhopadhyay, Resnick & Schauble, 1990, Cohen, 1965 and Leddy, 1977), and the "house in the sky" (positional) which has a stock of sand bags and balloons (opposites) affecting its position (Luth, 1967). As sand bags are purchased and put onto the house (addition), it becomes heavier, and moves down one foot per unit of the sandbag, while it rises one foot per unit when each sandbag is removed (subtraction). When balloons are attached, the house becomes lighter and rises one foot for each unit of the balloon, while it falls one foot for each unit of the balloon removed.

The following descriptions are of the physical interpretations of the directed number and neutralization models.

4.1 NUMBER LINE MODEL

An American survey (Crosswhite et al, 1986) found that (a) 87% of approximately 500 teachers of integers believed that "use of the number line helped in teaching integers" and that (b) "the formal approach to integers as equivalence classes was rarely used". For integer operations, 81% (addition) and 88% (subtraction) of teachers emphasized the use of rules rather than using the number line and physical situations. Ten recent grade seven textbooks (see Appendix B) were reviewed with respect to their approach to integer operations, and it was found that six of them used the number line exclusively, or in combination with profit and loss for example, for addition, while rules were used for subtraction. An additional two texts used the number line in combination with at least one other model, and only two did not refer to the number line for any integer operation.

4.1.1 Notions and Procedures

The procedure for using the number line to represent the set of **whole** numbers is as follows. Any number may be represented on the number line (marked off in equal intervals, closed at zero on the left, open at right) in one (or all) of three ways: (a) as a position (or address): for example 4 is located at the point labelled 4 on the line, or at the end of the vector starting at zero and ending at 4, as illustrated in Fig. 1 below.

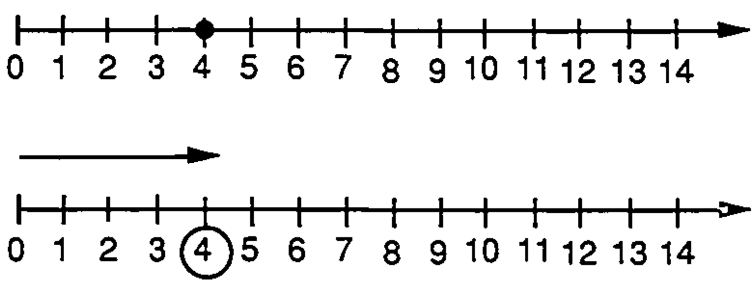


Fig. 1 Two representations of 4 as a position on a number line

(b) as a distance, for example the displacement from zero to the number labelled 4, or from any position to one 4 units further in any direction

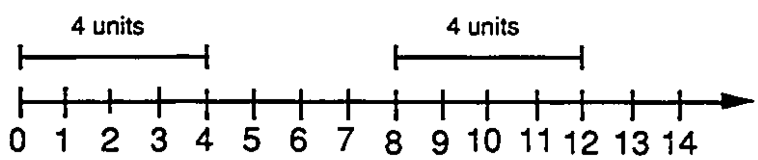


Fig. 2 Representation of 4 as a distance on a number line

(c) as a movement (jump), for example a move of 4 units to the right from zero, or 4 units to the right from any position, or 4 units to the left of any position greater than 4 (on the integer number line, of course, the movement can be 4 units to the left or right of any position).

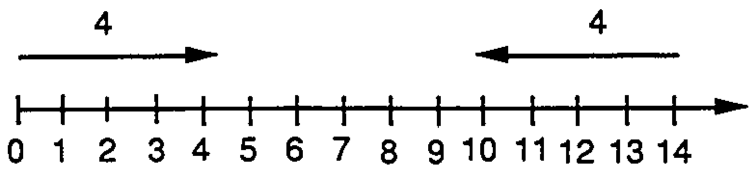


Fig. 3 Representation of 4 as a movement on a number line

For addition of two numbers, the leading number is located by a displacement from zero or by label on the number line, and then from that point, a movement to the right of the magnitude of the second number is made, as shown in Fig. 4 for a + b. The solution to the addition is the number at the position of the end of the second arrow (Fig. 4(a)) or is the length of the vector from zero to the end of the second arrow (Fig. 4(b)).

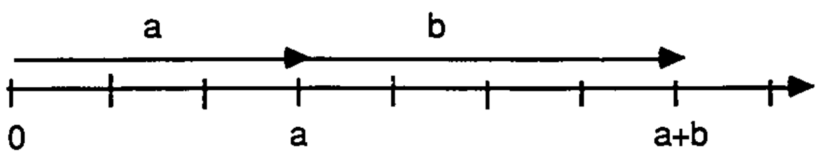


Fig. 4(a) Addition of a + b in N, solution found by position.

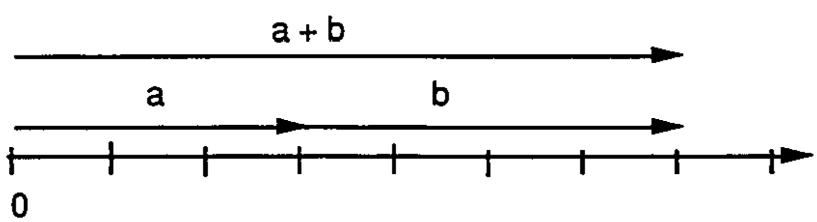


Fig. 4(b) Addition of a + b in N, solution found by vector.

Subtraction in N is a displacement to the left of a magnitude of the subtrahend, with the first number being located as in addition, and the solution identified as the position at the end of the last arrow, or as the distance from zero to the end of the last arrow, as illustrated in Fig. 5(a) and (b) for a - b (the magnitude of a necessarily greater than that of b):

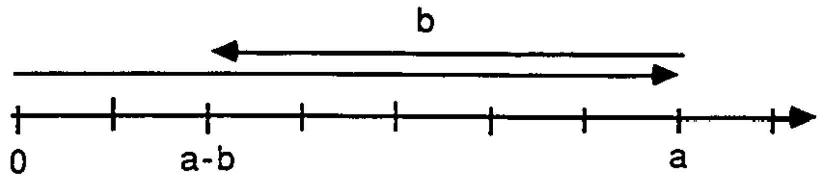


Fig. 5(a) Subtraction of a - b in N, solution found by position.

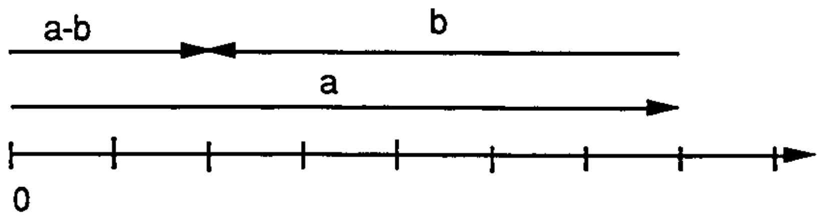


Fig. 5(b) Subtraction of a - b in N, solution found by vector.

When the number line is used for **integers**, the previous number line is usually extended from zero to the left, with natural numbers on the right of zero, and negative numbers on the left. This implies that $Z^+ = N$, but this is a false assumption, since N is embedded in (isomorphic to) Z^+.

For integer operations, the "rules" for the number line must change, as follows: One must still locate the leading number either by label on the line itself, or by position at the end of a vector displacement. If the number is positive, the displacement from zero is to the right, if it is negative, the displacement is to the left. For addition, if the second number is positive, the second displacement is to the right, if negative, to the left. For subtraction, it is assumed that in Z subtraction is the inverse of addition, and thus the direction of movements may be inversed (i.e. if subtracting a positive, move left (as in N), if subtracting a negative, move right). Fig. 6 illustrates the cases of pos + neg; neg + pos; pos - neg; neg - pos; neg - neg. (solution positions are circled).

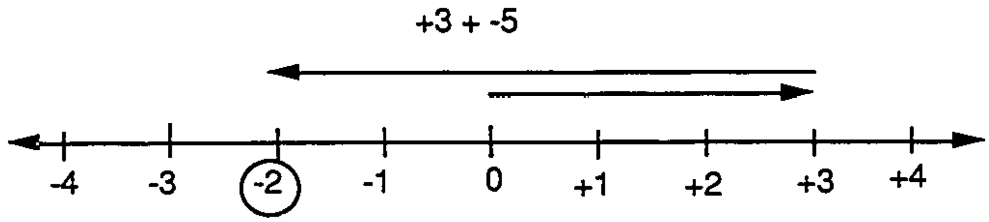


Fig. 6(a) Addition of a positive and a negative integer.

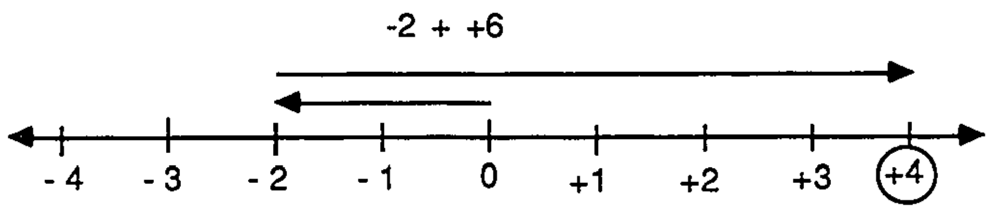


Fig. 6(b) Addition of a negative and a positive integer.

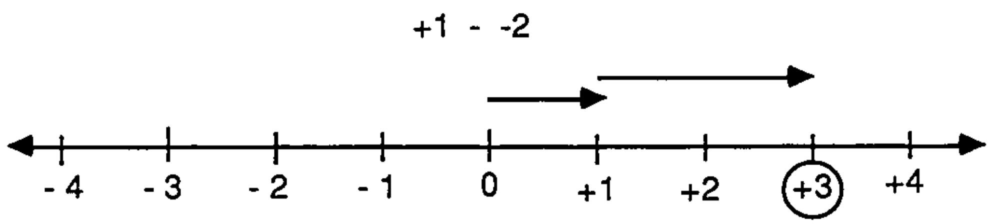


Fig. 6(c) Subtraction of a negative integer from a positive integer.

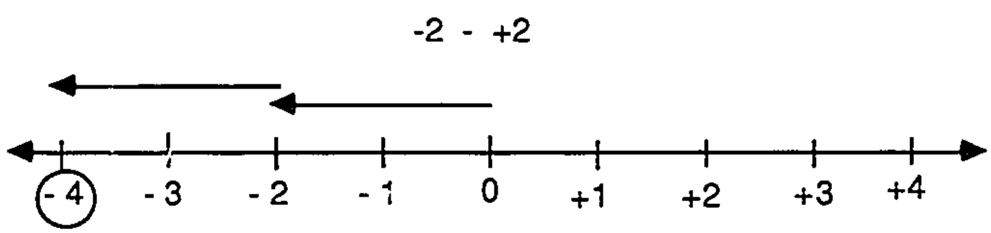


Fig. 6(d) Subtraction of a positive integer from a negative integer.

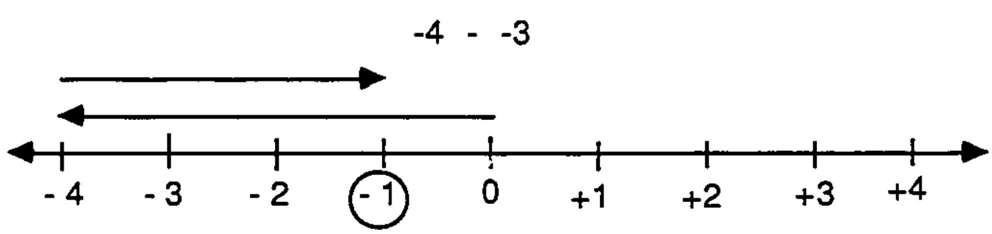


Fig. 6(e) Subtraction of a negative integer from a negative integer.

Many mathematics educators and researchers have advocated the use of the number line for integer operations for various reasons, a sample of which will be outlined below. Although only seven authors are quoted, it is reasonable to believe that their views are fairly representative of those who would advocate the use of this model.

4.1.2 Variations of Number Line for Integers

These authors use the number line for integers in the following ways:

[1] Magnuson (1966) uses it only as an extension of subtraction of natural numbers, allowing subtractions where the second number is larger than the first, for example 5 - 11. This is not an integer operation, but an operation on whole numbers which has an answer on the extended integer number line.

[2] Werner (1973) uses the conventional approach to integer addition, but for subtraction, she uses the number line to find the missing addend, because she feels that there are problems in representation with the conventional subtraction procedures on the number line. Thus she transforms the subtraction question (for example $^{-}6 - {}^{-}9 = \square$) to an addition one ($^{-}9 + \square) = {}^{-}6$), again assuming without proof that this manipulation that holds for N will also hold for Z. She first locates the known addend's position ($^{-}9$ in this case), then locates the position of the sum ($^{-}6$). An arrow is drawn from the sum to the addend, with an arrow indicating the direction to the sum. This displacement is the answer to the "missing addend", and hence to the subtraction. The solution to $^{-}6$ - $^{-}9 = \square$) (or $^{-}9 + \square) = {}^{-}6$) is shown in Fig. 7.

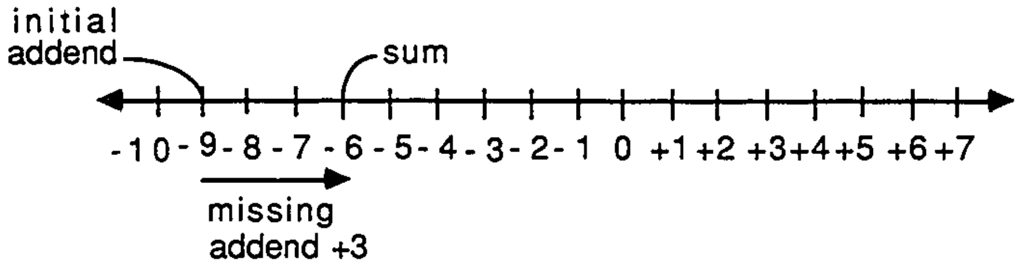


Fig. 7 Use of the number line to find the missing addend.

Ashlock & West (1967) use the same method (of finding the missing addend) but do it in a story context (for example "If I land at -5 and I have made a backward trip of four (-4), where did I start?", or "If I land at -5 when I started at -4, which direction did I travel, and how long was the trip?"). The operation is represented by the vector, and the initial and final positions are represented as such, just as for Werner.

[3] Hursh (1966) and Chang (1985) do not use the number line for subtraction, but change the subtraction to addition (use the additive inverse). The differences come in their explanations of this transformation. Hursh algebraically justifies subtraction as the addition of the opposite as $a - b = a + (-b)$, yet at this stage (i.e. for the student first being exposed to integers), surely this is an assumption since this notion of subtraction does not exist in N. He also allows the definition of $a - b = c$ iff $a = c + b$ to warrant the subtraction procedure of finding the additive inverse, then uses the number line as Werner does. Chang immediately changes the subtraction problem into an addition one ($^-4 - {}^-2 = {}^-4 + 2$) with no explanation given for the transformation, then performs the addition on the number line.

[4] Chilvers (1985), Coon (1966) and McAuley (1990) all use the conventional procedures discussed above for both addition and subtraction on the integer number line (use subtraction as an inverse operation). McAuley wants to find a procedure for getting $^+2$ as the result of $^+5 - {}^+3$, and to get this, he must do the opposite movement to that of addition, which is a "rule" of the minus operation. For Coon, the sign of the number indicates the direction of the vector, and subtraction necessitates a move in the opposite direction from

addition since it is an inverse operation. Addition of a negative should be an addition that occurs in the negative direction.

Chilver's justification for the direction of displacements is more concrete for the student, but also uses the notion of subtraction as an inverse operation. He uses the image of a person facing right from zero at the beginning of the exercise. If he must displace a positive number, he moves forward; a negative, backward. Thus the sign of the number indicates direction. If addition is the operation, he faces right, if subtraction, left. Thus his thinking for $^{-}5 - {}^{+}7$ would be as follows:
The first number is negative, so he moves back 5, the operation is subtraction, so he must face left, and moves forward 7, since the second number is positive. This is illustrated in Fig. 8. Although these manipulations result in a correct answer to the integer task, they are not rationally based.

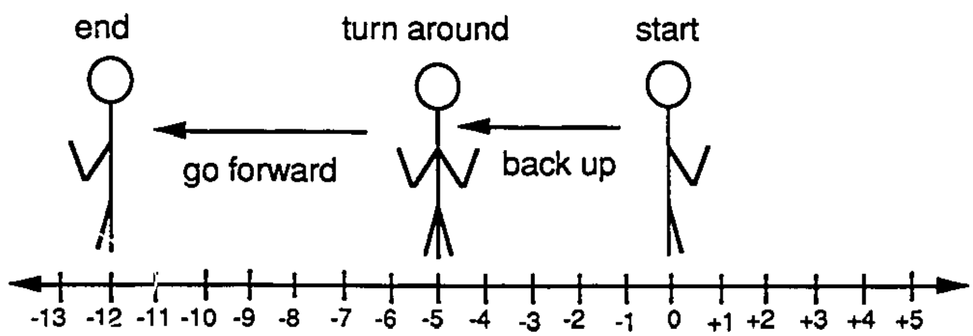


Fig. 8 Chilver's use of the number line for $^{-}5 + {}^{+}7$.

4.1.3 Motivations for Using the Number Line

The following are given as motivations of these authors for the use of the number line as a model for integer operations:
[1] It is a model which can be used for all four operations on integers (Chilvers, 1985). He has developed procedures that "make sense" of addition, subtraction, multiplication and division using the number line. This does not seem to be a valid argument, since although he is consistent in his use of signs of numbers and of operations for addition and subtraction, he changes this convention for multiplication and division, where the sign on the second number

still indicates which way to move (forwards or backwards), but the sign on the first number now really becomes the operation sign (which way to face - left or right), while the numeral becomes a whole number rather than an integer. Multiplication becomes repeated addition or subtraction, and division, as the inverse of multiplication changes from a question of $^+2 \div {}^+3$ to $^+2 \times {}^+1/3$, which depends on the student knowing reciprocals, and interpreting $^+3$ as the fraction $^+3/1$, (or $^+3/^+1$?). It would seem that he has not considered all of the implications of extending this model into multiplication and division.

[2] <u>It is a model which can also be used in other number systems</u>, notably N, which the student encountering integers will be familiar with, but also Q and others, and one which is therefore adapted to the learning level of the child (Coon, 1966). He expresses concern that the use of the model for N <u>should not create contradictions when extended to other number systems</u>, and Werner's (1973) adaptation of the number line stems from her desire for a "smooth transition" from N to any other number system to be provided by the model. Coon therefore does not want subtraction in N to be represented as a backwards jump, but rather as a move in the direction opposite to addition, thinking of the procedure that will be used later for vector operations. Unfortunately, his procedure for multiplication is the same as that of Chilvers, described in point [1] just preceding, which does not preserve the meaning of the signs within the model. Werner's solution is to interpret subtraction in terms of addition by finding missing addends.

[3] It is <u>effective</u> (Werner, 1973), as it has been "the means to guided discovery for many pupils", yet even as Werner states this, she expresses a concern about the "limitations" of the number line for subtraction beyond the set of natural numbers, and feels that she must propose changes in its use. Hursh (1966) is also convinced of its effectiveness "in making (many) topics of mathematics more teachable and understandable".

[4] It is "an <u>intuitive</u> development" of a situation involving signed numbers, (Magnuson, 1966). It is a "<u>natural</u>" way to "present, display and discuss properties of the real numbers" (Hursh, 1966).

[5] It is practical - it "answers the 5 - 8 question" (Magnuson, 1966). This seems to be the only type of question that Magnuson concerns himself with, as he never uses it with integers, only with the subtraction of whole numbers.
[6] It is a device with the "potential" for "conveying mathematical ideas quickly and accurately" when one has so much to teach in the available time (Hursh, 1966).
[7] It "gives visual assistance" in performing operations on integers and in making sense of results. It also involves mechanical motion which gives a feeling for the action of operations (Hursh, 1966). This argument leads one to wonder what conception Hursh has of number operations, particularly of subtraction, which should convey some notion of "taking away" (it can be argued that the number line motion for addition is indeed "combining" two displacements). None of the versions of this model use the notion of take-away subtraction, and Hursh himself "invents" a motion for this operation that he justifies algebraically (i.e. abstractly).

4.1.4 Concerns About the Use of the Number Line

These authors also express some concerns and cautionary remarks about the number line, as follows:
[1] It should not be used in isolation, and should be included with other models (Magnuson, 1966).
[2] Because it is a good model for future number systems, the vector approach must be used (Coon, 1966).
[3] Subtraction is a problem (Werner, 1973), and especially the use of the "backwards jump" becomes a problem for subtraction of negatives (Coon, 1966).

Several other authors have expressed concerns about the use of the number line as a model of arithmetic operations, even in the set of natural numbers. Again, even though only a few articles are cited here, their ideas seem to represent those of others. The following concerns have been expressed:

[1] The number line is "highly symbolic" (Carr & Katterns, 1984, Rathmell, 1980), "arbitrarily representational" (Ernest, 1985), a tool which provides procedures to obtain answers, but does not necessarily promote understanding of the numbers or of the operations (Fuson, 1984).

[2] Inherent within the model is the confusion between number as measure (vector or unit measure) and number as count (label) (Carr & Katterns, 1984). Thus when a child counts by hopping a certain number of units, then uses a point as the solution, one must consider what the child understands about the meaning of the answer (Fuson, 1984). Semadeni (1984) objects to the change of state from "static" to "dynamic" then back to "static" since it identifies numbers with functions. (One could comment here that Werner's missing addend approach uses the point notion to locate the two given numbers, and the vector to represent the solution.) Fuson proposes a "support" to the use of the number line (a second model to assist understanding of a model to assist understanding of operations on numbers!). She would use a device such as Cuisinaire rods lined up along the line to indicate that the solution is really a measure solution, consistent with the measures used in the problem (presumably indicating the solution as a vector from zero would serve the same purpose).

Fischbein (1977) states that the inconsistency between the notion of number on the number line as movement or as position becomes heightened when the numbers go beyond the set of integers. He acknowledges that 1/2 can be represented by half a move, but that for other rational numbers "we still prefer points on the number line instead of moves".

[3] We have no proof that the number line promotes understanding (Ernest, 1985).

[4] While the version of the number line described by Chilvers (1985), which employs a man turning and moving, is the only one that works for both addition and subtraction using the same set of constraints, Bell (1983) maintains that with this interpretation "subtraction has ceased to be an operation combining two numbers", and that "the pupil is cut adrift from all his existing intuitions about the meaning of numbers and operations". Presumably he would

include addition in this vein, as both the operations sign and the sign of the negative number become operators of movements and are separated from the number itself. These artificial movements are also not related to the way negative numbers are used in any actual situations.

[5] There is a concern that large-scale survey assessment items which require number line responses may not measure mathematical operations, but only number line competence (Ernest, 1985), as the number line itself as a model must be mastered before it may be used competently for number operations.

[6] "Many children do not understand the use of the number line as a model for addition" (Rathmell, 1980). This comment was based on the analysis of errors for a number line drawing (natural numbers) requiring the student to write a matching sentence on the NAEP 2 assessment survey. Rathmell found that both numbers at the end of the arrows as in Fig. 1 (i.e. a and a+b rather than a and b) were used as addends by 45% of the 9 year olds and by 39% of the 13 year olds surveyed. He reports that the same survey included a multiple choice addition task modelled by joining two sets, and success rates for this task were much higher (as seen in Table 2 below). Although there was no testing of subtraction using sets, he reports that only 14% of the 9 year olds and 33% of the 13 year olds were able to correctly write a subtraction sentence when given a number line illustration of subtraction. Rathmell was led to conclude that the number line model for any operation is not well understood, and needs recognition of each unit along the line as an object.

Table 2: Success on NAEP2 Addition Tasks

MODEL USED	9 yr. olds	13 yr. olds
diagram of two sets being joined, select open number sentence from those provided	74 % selected the addition sentence	90 %
from a number line marked with arrows, form a number sentence	25 %	48 %

4.1.5 Implications for the Present Research

Carr & Katterns (1984) feel that the inadequacy of the number line lies in the way that it is taught, and recommend an improvement in teaching to overcome the problems. Bell, Costello & Kuchemann (1983) state that for subtraction, "models are complicated, and rules are very likely to be confused and misused. Many secondary school children are not going to be able to cope with subtraction in an abstract way, and at the same time they are not provided with an accessible model". Kuchemann (1981) advocates abandonment of the number line for integer operations in favour of an annihilation model.

In view of the above (i.e. the preference of the number line as a model to teach integer addition and subtraction, the criticisms of such a model, and the lack of success of students with these operations when taught using this model), it was decided to study the use of an approach for integer addition and subtraction that models both operations physically, using the notion of **neutralization**.

4.2 THE NEUTRALIZATION MODEL

The neutralization model is a concrete model which usually encourages physical manipulation, if not of actual colored bingo chips or squares of paper (each representing either a positive unit or a negative unit which are opposite to each other), then of positive and negative signs written on paper. It is consistent with addition and subtraction in N, in that addition physically means to add a quantity and subtraction physically means to take away a quantity.

There are a few variations of this model in the literature, and in fact, no description has been a complete one, but there are several features in common to each, and a composite picture of this model can easily be constructed. In contrast with the various ways in which operations (and indeed integers) can be interpreted using the number line model, no description of the neutralization model has been found to conflict with any other description, and operations using the model are the same for each author. The only inadequacy is that some authors have not realized the full potential of the model, and have used it in a limited way. Thus the following description is this author's overview of the articles which advocate the use of the neutralization model in some form.

4.2.1 Notions and Procedures

Simply stated, the basic concept is that positive and negative numbers are **opposites**, and that an equal quantity of positives and negatives **neutralize** (annihilate or cancel) each other to **make zero** (the neutral element). Thus zero represents the **equivalence class** of all pairs of equal amounts of positives and negatives, as $^{+}3 + {}^{-}3$, $^{-}18 + {}^{+}18$, etc. are all equivalent to zero.

The term "neutralize" was chosen here for the following reasons:

(a) the word "cancellation" brings with it various associations which are not related to the concept being focused on. Children will be familiar with cancelling a game due to rain, school due to a snowstorm, an appointment, a subscription to a magazine, etc. In mathematics, when one cancels in either the subtraction or fraction multiplication algorithms, one "crosses out" (Oxford) and either regroups or eliminates a common factor, but does not obtain a resultant value, in our case the neutral element zero.

(b) the term "annihilation" is often used in terms of "utter destruction" (Oxford), which conflicts with our notion of the resultant neutral element zero. Children, unfortunately, may have this notion from gangster-type broadcasts.

(c) Children will be familiar with the notion of being "neutral" in a game or a discussion, which means to not take sides - to be in the middle, to be "safe" in a neutral zone between two opposing teams or views. They may also encounter the neutralization of acids and bases in their science classes (although here neutrality is at pH 7 rather than zero!). The Oxford Concise English Dictionary defines the verb neutralize as "to render ineffective by opposite force or effect", which fits the mathematical statement $^{+}n + {}^{-}n = 0$.

(d) Both cancellation and annihilation describe an action, while neutralization describes both the action and the effect (i.e. of neutrality).

For these reasons, the primary term used will be "neutralization", with "cancellation" and "annihilation" possibly used to support the notion of the action of rendering neutral.

A particular integer may have many different physical or numerical representations, due to the non - influence of the presence of the **neutral element** zero, which itself may be represented in an infinite number of ways, since any $^{+}n + {}^{-}n = 0$. Thus any integer designates an **equivalence class**, each member of the class having the same **resultant value**, determined by the value of any quantity left over when all neutralized elements are constructed (m represents all $^{+}n + {}^{-}n + m$). A positive number has an abundance of positives, and a negative number has an abundance of negatives. For example, $^{+}7 = {}^{+}7 + {}^{-}2 + {}^{+}2 = {}^{+}7 + {}^{+}8 + {}^{-}8$, etc. and $^{-}2 = {}^{-}2 + {}^{-}10 + {}^{+}10$, or in the ordered pairs notation, $^{+}7 = (7,0) = (9,2) = (15,8)$ and $-2 = (0,2) = (10,12)$, etc.

When determining a sum after combining the two addends, it may be necessary to "make zeros" and use the quantity left over, i.e. the resultant value, as the answer. When doing subtraction, it may be necessary to first "add zeros" to the minuend to ensure that there are enough positives or negatives to be taken away.

The simplest, clearest, description of the model is found in Grady (1978) and Freudenthal (1983), while Bennett and Musser (1976) provide more detail. The opposites positive one and negative one are represented by different symbols (Grady uses white squares for positive, dark squares for negative). The use of different colors for positive and negative numbers occurs as early as 200 BC, as the Chinese were known to use red computing rods for positive numbers, and black rods for negative ones (Smith, 1925). The value of any particular group of squares may be determined by first neutralizing i.e. isolating all combinations of opposites to form neutral elements, then by counting the remaining squares. Thus the grouping shown in

Fig. 9 containing 7 black and 4 white squares contains 4 matches of opposite units, and the **resultant value** of the grouping is 3 black, or $^{-}3$.

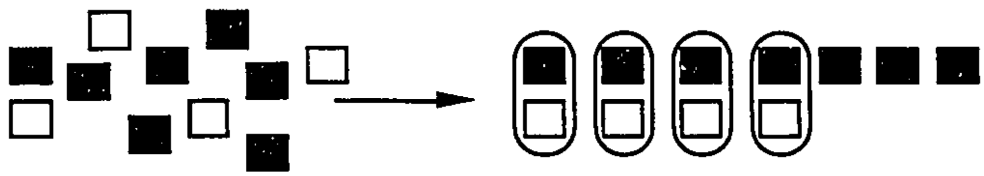


Fig. 9 Determination of resultant value of a group by matching equal amounts of opposites.

Integer **addition** falls into two classes - addition of integers of like signs, and addition of integers of unlike sign. To add integers with like signs, e.g. $^{-}2 + {}^{-}7$, one simply puts out 2 dark squares, then adds 7 more dark squares to the first group to get a total of 9 dark squares, or $^{-}9$, as illustrated in Fig. 10. This requires a trivial procedure of putting together and counting, not differing from the addition procedure in N.

Fig. 10 Addition of chips of like sign ($^{-}2 + {}^{-}7$) by combining sets.

When the signs are different, for example $^{-}5 + {}^{+}3$, one puts out 5 dark squares and 3 white squares, matching as many opposites to make "zero" as possible, in this case 3. The resultant 2 dark squares represents $^{-}2$, the sum of the two integers, as shown in Fig. 11. This addition procedure is more demanding than the first, requiring a search for a canonical value through neutralization after combining groups.

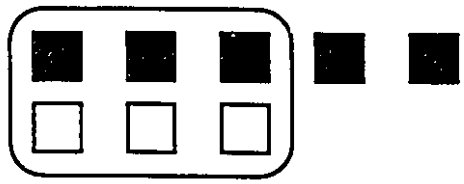


Fig. 11 Addition of chips of unlike sign ($^{-}5 + {}^{+}3$) by combining sets and isolating the neutral element.

For **subtraction**, there are also two possible cases: either (a) there will be enough to subtract (this type will be referred to as <u>trivial</u>, since the procedure is no different than that in N), or (b) one has to construct a representation equivalent to the first number, containing a sufficient number of chips to allow for removal of the second amount. An example of the first case would be $^{-}9 - {}^{-}2$. Nine dark squares would be placed on the surface, then 2 of them would be taken away, leaving 7 dark ones, or $^{-}7$.

Fig. 12 "trivial" subtraction of $^{-}9 - {}^{-}2$

The second case is illustrated with $^{+}4 - {}^{+}7$, where the integers have the same sign, and with $^{+}3 - {}^{-}2$, where the signs are opposite. In both examples, there will initially not be enough squares to take away, so one must add groups of "zeros" until there are enough. Chang (1985) always adds a number of groups corresponding to the amount to be subtracted, since any extra zeros will not affect the outcome, and perhaps simplifies the decision of how many zeros to add. The two examples given above are illustrated in Fig. 13. This type of subtraction is the most demanding of all the addition and subtraction procedures, since it is not only necessary to change the representation of the initial amount, but also to determine which particular representation is required for the particular subtraction task.

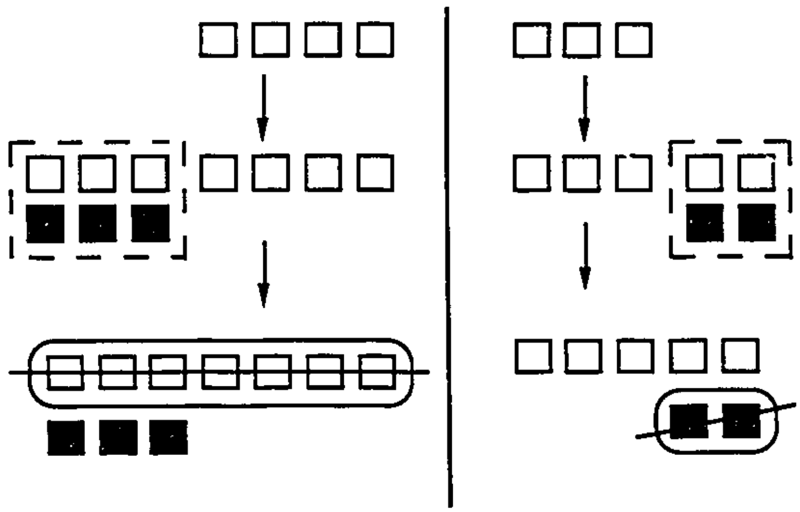


Fig. 13 Subtractions requiring the addition of the neutral element ($^{+}4$ - $^{+}7$ and $^{+}3$ - $^{-}2$)

Cotter (1969), Battista (1983), Kohn (1978), Sherzer (1969) and Whimbey and Lochhead (1981) advocate a transition from the concrete material to paper and pencil, drawing plus and minus signs to represent the two types of integers, using the correct amount of signs ($^{+}3$ will be represented by 3 plus signs). Thus, $^{-}2$ + $^{+}7$ would be represented in Fig. 14.

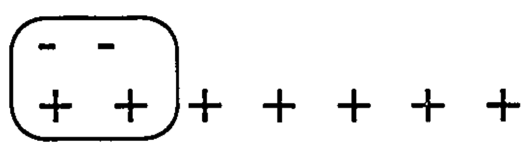


Fig. 14 Use of symbols to represent $^{-}2$ + $^{+}7$

While **multiplication and division** using the model are beyond the scope of this thesis, it must be stated that these operations are interpreted in this model and others as a whole number operating on an integer value, and cannot be considered to be a valid use of this model for integers. For example, $^{+}3$ x $^{-}6$ is interpreted as (adding) 3 groups of 6 negatives (because the leading number is positive), and $^{-}2$ x $^{+}5$ is removal of 2 groups of 5 positives (because the leading number is negative). Division is interpreted as the inverse of multiplication, and the procedure is to find the missing factor.

Most authors expect that either the student will discover the equivalence of adding an amount with subtracting its opposite (Jencks & Peck 1977, Gibbs 1977, Dirks 1984, Cotter 1969, Kohn 1978), or expect the instructor to draw this to the child's attention in some way (Bennett & Musser 1976, Whimbey & Lochhead 1981) (by setting up examples, or by a direct statement) in order to formulate the subtraction rule "to subtract, add the opposite", while others (Bartolini 1976, Battista 1983, Grady 1978) do not concern themselves with rules, but remain with the concrete model. Fremont (1966) is the only one who gives the student the freedom to make up his or her own rules.

4.2.2 Variations of the Neutralization Model

Variations of the concrete representation of the neutralization model use different embodiments of opposites, but are fundamentally the same, although not all authors extend the model to subtraction or to symbols or mathematization. These variations are:

a) positives and negatives

Haner (1947) seems to have initiated the use of the neutralization model as a teaching tool, describing a region of equally matched positives and negatives as a "region of deadlock", likened to a draw card pile or bone pile in card games (which have since been replaced by other amusements, and will not be so familiar to today's child). Any unmatched positives or negatives are the "active surplus" and give the integer its value (Fig. 15(a)).

```
+ + + + +       + + +
- - - - -
```

region of deadlocked positives and negatives

active surplus of 3 positives

Fig. 15(a) a symbolic representation of positive 3

He uses the model for subtraction only, borrowing from the deadlocked area, which frees the signs of the opposite kind (indicated by a *) when the active surplus does not contain enough to subtract (Fig. 15(b)).

```
      *   *
+ + + + +       + + +
- - - [ - - ]
```

Fig. 15(b) Subtraction of negative 2 from positive 3.

He illustrates the equivalence of operations by also matching some of the deadlocked positives or negatives with the addition of the opposite of the subtrahend, again releasing more into the active surplus, as shown in Fig. 15(c).

```
      *   *
+ + + + +    + + +
- - - [ - - ]
      +   +
```

Fig. 15(c) Addition of positive 2 to positive 3

b) <u>charged particles</u>

Cotter (1969), who claims to have constructed the neutralization model himself, uses the notion from chemistry of charged particles, where a positive and a negative particle are attracted to each other. He calls this neutralization of charges, and labels it zero. He uses the concept of resultant charge, and deals directly with the notion of equivalent operations by asking "Is there any other way that you could begin with a zero charge and end up with a charge of positive four?". Instead of a subtraction rule, he arrives at a "definition" of subtraction in terms of addition ($a - b = a + {}^{-}b$). Cotter's description is the most complete use of the physical model, omitting only the explicit reference to opposites. He uses the model with multiplication and division as described on page 45.

Kohn (1978) also uses the notion of positive and negative charges, and uses colored chips to represent them. She follows the procedures described above for neutralization, resultant value, addition and subtraction, and expects that students will derive the subtraction rule from having to add the extra zeros in subtractions, yet she begins with a representation of the initial number which contains enough zeros to allow for the subtraction. She uses the model for multiplication as described on page 45.

Whimbey & Lochhead (1981) also use the model with charged particles, using symbols rather than physical "chips". They use the notions of cancellation and net charge, and avoid the addition of zeros for subtraction by having enough extra signs displayed for the initial number. They formally introduce the rules after the students have written mathematical sentences for pictures which show three boxes - one with the initial number, one with either the number to be added or to be subtracted, and the final box containing the resultant number of charges (the students do not have to actually do the adding or subtracting, just identify the charges on all three boxes - although they speak about performing operations simultaneously with the charged particles and with integers). Their work has been with remedial students, and their concern is that the students have a model to help them learn the rules (i.e. that they have a concrete illustration of the rules).

Battista (1983) puts positives and negative chips into jars, and evaluates the "charge on a jar" by the cancellation of positives and negatives. He stresses the equivalence class of any integer, where integers are "collections of charges in jars". Addition and subtraction are as defined above, and he uses the model for multiplication and division as described on page 45.

c) <u>hot and cold cubes</u>

Jencks & Peck (1977) interpret the notion of opposites in an artificial situation of a witch's brew of hot and cold cubes, each representing a change in temperature (lowering or raising) of one degree. Neutralization is discussed, but not directly used in the model, as only reasoning skills are referred to in determining the results of addition and subtraction (if the temperature is $^{+}4$, and 10 cold cubes are added, the temperature will be lowered by 10 degrees to $^{-}6$; the removal of cold cubes makes the temperature rise, etc.). There is also some confusion about notation, as in $^{-}3 - {}^{+}5 = {}^{-}8$, $^{-}3$ and $^{-}8$ are described as representing temperatures, and $^{+}5$ denotes 5 hot cubes. Students should "discuss and convince each other" of correct results, rather than carrying out a mathematical procedure either on the cubes or on numbers. They expect students to discover the subtraction rule, which may be easier with this version of the model, since the emphasis is on reasoning what happens to the temperature when cold cubes are subtracted and hot cubes are added.

d) <u>holes and plugs</u>

Gibbs (1977) uses holes (H) and plugs (P) which fit together exactly to make a blank, as shown in Fig. 16.

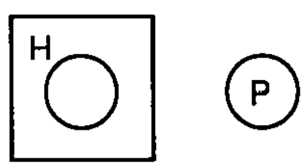


Fig. 16 One hole and one plug

The notions of opposites and neutral element which has no effect on values are used. Addition is "contribution" of holes or plugs, and subtraction is "removal" of holes or plugs. Number sentences are written with H's and P's designating holes and plugs, and no reference is made to positive or negative until operations are completed with the physical model, and this step is left to our imagination. It is expected that the student will generalize the rule for subtraction. For multiplication, Gibbs turns to a tabular format, based on the results of multiplication of holes and plugs performed as described on page 45.

e) loops facing left or right

Fremont (1966) uses pipe cleaners twisted into a curved shape (because a curve to the right resembles the curve on a P, it is used for a positive number; since negatives are opposites to positives, they are curved to the left). He uses cancellation of opposites, demonstrating this on a number line, and shows that the two curves put together resemble the shape of zero. This is shown in Fig. 17.

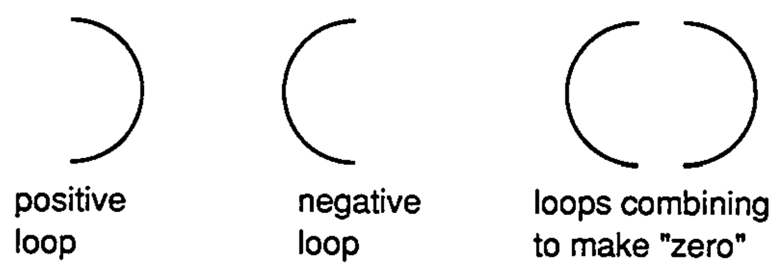


Fig. 17 Pipe cleaner loops

He feels that larger number examples will be beneficial to force the student away from using the physical model so that he will discover a rule that he can use to perform the operation. The physical model can then be used to check the child's conjectures. Addition and subtraction are performed as described above. He advocates using addition to check subtraction results, but the reversibility of addition and subtraction has not yet been proved in Z for the student.

f) integer abacus

Bartolini (1976) cuts a number line in half to set up an integer abacus, one line representing negative values, and one positive. When there are an equal number of circles on each line, zero is represented (implicitly indicating the equivalence class of zero). Adding and subtracting are as described above, with a line being drawn at the zero level after each operation, and the quantity above the line indicating the resultant value. Fig. 18 shows the initial abacus, and the abacus after performing $^{+}3 - {}^{-}2$, which entails adding 3 positive circles, and removing 2 negative circles. The author does not "add zeros" for subtractions, as the abacus always has extra circles below the zero line.

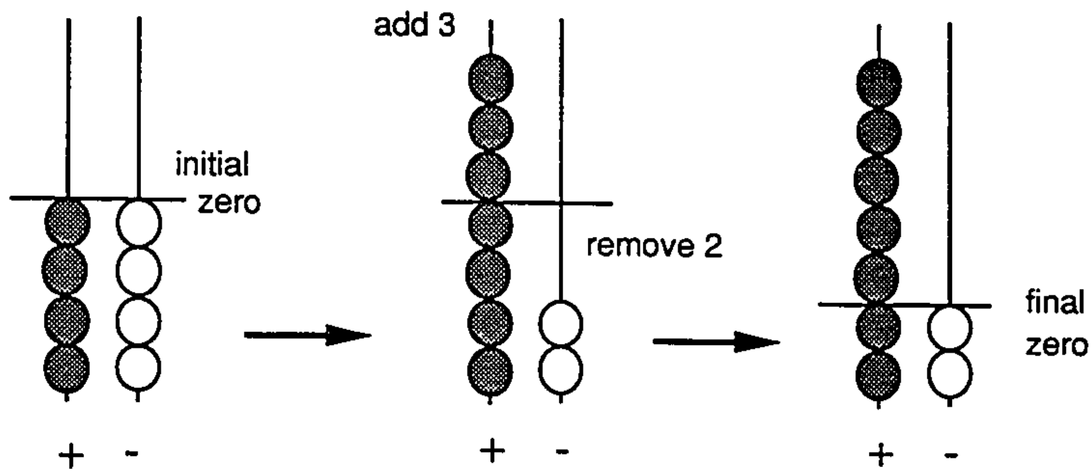


Fig. 18 The integer abacus used for $^{+}3 - {}^{-}2$

Dirks (1984) uses the same abacus representation as Bartolini, but stresses that the abacus is only a tool, and should begin to serve as a mental model for operations. He expects the subtraction rule to develop from the model. He uses the abacus for multiplication as described on page 45.

(g) <u>scores and forfeits</u>

Liebeck (1990) uses colored counters, each counter representing either a score of one British pound, or a forfeit of one pound, and operations on the counters are put into the context of a "game" situation (for example: "you won a treasure hunt: 3 scores"). Neutrals are never removed, leading to an awareness of equivalence classes of the resultant value, and only 5 of each type of counter are allowed per student, forcing the problem situation, for example, of how to add another score to a group of 5 scores and 3 forfeits, thus leading to the equivalence of operations (subtract a forfeit to get the same result). Students write number sentences to represent each operation, using their own notation for forfeits.

h) $^{+}$1 and $^{-}$1

Sherzer (1969) uses a more symbolic version of the neutralization model, and just writes $^{+}1$ and $^{-}1$ to represent the decomposition of integers (for example, $^{-}4$ is written as $^{-}1$ $^{-}1$ $^{-}1$ $^{-}1$). He explains that $^{+}1 + {}^{-}1 = 0$, and finds results of addition as described above. He expects addition rules to develop from the model. He uses set theory and ordered pairs as justification for the use of the model, and uses ordered pairs to perform additions. He does not attempt to use the annihilation model for subtraction.

i) ordered pairs

The use of ordered pairs to represent integers and their operations will be discussed more in a more compete form shortly, but in this section there will be a description of how certain authors have interpreted this notion and used it as a teaching tool.

Tobias et al (1982) use the more abstract notion of ordered pairs to represent integers, which they develop from a convoluted arithmetic exercise of finding "cross-tots". Equivalence classes and canonical form (families of the smallest number pair) are stressed, and in fact even graphed. Addition is performed only on ordered pairs, and later integer notation is introduced and the ordered pairs become the mental model. The neutralization statement becomes an addition of two ordered pairs (0,5) + (5,0) = (5,5) = (0,0). Subtraction is performed by finding the missing addend of the ordered pair, which thus produces many answers (all members of the same equivalence class), and resembles a guess and check procedure. Changing subtraction into the addition of the inverse was a second strategy suggested. Their series of articles are reports of a teaching experiment, and their focus was not on integer notation, but on manipulation of "cross-tots" and ordered pairs.

Baily (1974) talks about the "mutual annihilation" of red and blue counters, and for addition does not mention mathematization, but for subtraction, he uses ordered pairs (as symbols of a real-life situation of football scores) to perform the operation.

Bélisle (1989) uses ordered pairs in a board game situation to determine outcomes of rolls of 6 dice, each having 3 faces colored red, 3 faces green. After determining resultant values of (4r,2g) for example, the student moves his token accordingly. Neutralization occurs with (3r,3g) where the student does not move a token, and when all of the possible outcomes are compared, ordering may be determined based on best and worst move combinations (Bélisle is the only person who orders using the neutralization model). Cards showing different combinations of ordered pairs are matched based on equivalence, and integer notation is used to record the resultant value, so that additions of ordered pairs (related mentally to the game situation) may be translated into integer notation. For subtractions, an equivalent ordered pair is used for the leading number.

4.2.3 Mathematization of Integer Operations from the Neutralization Model

Mathematization from a model means that one should be able to make meaningful mathematical (in this case arithmetical) statements based directly on the workings of the model, and once these statements are made, reliance should shift away from the model and to the mathematical realm. Operations are no longer performed on concrete material, but on numbers. As mentioned earlier, Greeno (1991) states that the same manner of operations within the physical model (combining and separating) should be used to construct equivalent operations within the mental model of integers. This process seems to be difficult when the model is the number line, with operations as "shifts" along the line, as for example the procedure for subtracting a positive integer is always the same as the procedure for adding a negative integer (by definition within the model), so the mathematical statement $x - y =$

z covers both situations, neither one uniquely, and in fact y is treated as a whole number rather than as an integer. The only version of the number line model that does not have this particular problem is that of Chilvers (1985), where subtraction is always the opposite of addition.

Very few authors seem to be concerned with the mathematization of an integer model, seeming to be content to use the model purely for concrete manipulation, or to formulate rules directly. However, there is some evidence of two approaches to this extension of the neutralization model, namely the <u>decomposition and cancellation</u> approach, and the <u>equivalence of ordered pairs</u> approach. Both of these approaches rely on changing the representation of one of the integers before some additions and subtractions may be performed, and this change is developed from the physical model. Bell (1983) calls this a "disadvantage" of the model, but this restructuring is consistent with known subtraction algorithms in both N and Q, as illustrated below, and in fact, the operation of subtraction in all three number sets thus requires the transformation of the minuend into an equivalent representation.

<u>Subtraction in N</u>: 64 minus 17 requires that one changes the representation of 64 as 6 tens and 4 ones to an equivalent one of 5 tens and 14 ones before one is able to subtract the 7 ones, as follows:

$$\begin{array}{r} 64 \\ \underline{-17.} \end{array} \quad \text{becomes} \quad \begin{array}{rr} 5 & \\ \cancel{6} & 14 \\ \hline -1 & 7. \end{array}$$

<u>Subtraction in Q</u>: The task $2\frac{1}{4} - \frac{3}{4}$ requires that the representation of $2\frac{1}{4}$ is changed from that of 2 wholes and 1 quarter to an equivalent one of 1 whole and 5 quarter before 3 parts can be subtracted, as follows:

$$2\frac{1}{4} - \frac{3}{4} \quad \text{becomes} \quad 1\frac{5}{4} - \frac{3}{4} = 1\frac{2}{4}$$

a) DECOMPOSITION AND CANCELLATION

There are only two authors who use the notion of decomposition (in order to perform cancellation of zeros) of one of the numbers involved in addition or subtraction as a strategy for the mathematization of this model, and even at that they see this step as just a detailed version of the operation performed. Cotter (1969) gives one example from addition (where the number with the largest absolute value must be decomposed):

$^{+}7 + {}^{-}5 = ({}^{+}2 + {}^{+}5) + {}^{-}5$	renaming
$= {}^{+}2 + ({}^{+}5 + {}^{-}5)$	associativity
$= {}^{+}2 + 0$	neutralization
$= {}^{+}2$	evaluation

Bennett and Musser (1976) use a subtraction example (where the leading number must be re-expressed in an equivalent form):

$$5 - 8 = [5 + 0] - 8$$
$$= [5 + (3 + {}^{-}3)] - 8$$
$$= [(5 + 3) + {}^{-}3] - 8$$
$$= [8 + {}^{-}3] - 8 = {}^{-}3$$

The addition decomposition follows the procedure used for subtractions in N and Q (decompose the larger number), while the addition of opposites for subtraction requires bringing in extra numbers.

b) EQUIVALENCE OF ORDERED PAIRS

The ordered pairs approach to integer operations, where the first number in the ordered pair represents positives, and the second number negatives, is based on set theory, which was popular in the sixties and seventies. The group (Z,+) is an additive group, and subtraction is not a defined operation, but when ordered pair notions are interpreted here in the context of particular "concrete" numbers, this operation has a definition governed by the corresponding action on the concrete material.

The canonical representation of a positive integer n is expressed as (n,0) and a negative integer as (0,n), for example, $^{+}6$ is (6,0) and $^{-}12$ is (0,12). Resultant values are obtained from (m,n) when m > n by (m-n,n-n) and when n > m by (m-m,n-m) to get zero as one element of the pair (thus (9,4) = (9-4,4-4) = (5,0) and (2,11) = (2-2,11-2) = (0,9)). Sherzer (1969) and Tobias et al. (1982) define the notion of equivalence as: (a,b) is equivalent to (c,d) if a + d = b + c (thus (5,4) is equivalent to (9,8) since 5+8 = 4+9). When rearranged in the form a - b = c - d for a > b and c > d, or b - a = d - c when b > a and d > c, this gives the criteria for equivalence as resultant values being equal (thus (5,4) is equivalent to (9,8) since they both reduce to (1,0), the canonical form). Galbraith (1974) as well as Tobias et al (1982) use the concept of "sets of ordered pairs" which are equivalent because they are representations from the same class of integers, for example all members of the set (2,0) (3,1) (4,2) ... (m+2,m) are represented by the conventional integer notation of $^{+}2$. Addition is defined to be the addition of the corresponding parts, i.e. (a,b) + (c,d) = (a+c,b+d), where the result is then expressed in canonical form, i.e. where one element in the pair is 0. For example, $^{-}2 + {}^{+}3$ becomes (0,2) + (3,0) = (0 + 3,2 + 0) = (3,2) = (1,0) = $^{+}1$. This parallels the removal of "zeros" in the physical model. Baily et al. (1974) demonstrate this approach for subtraction, i.e. (a,b) - (c,d) = (a-c,b-d), where it may be necessary to change the representation of the first integer to an equivalent one so that there will be enough to subtract (i.e. a-c > 0, b-d > 0). For example, $^{-}3 - {}^{+}4$ = (0,3) - (4,0) = (4,7) - (4,0) = (4 - 4,7 - 0) = (0,7) = $^{-}7$. This parallels the addition of "zeros" in the physical model to change the representation before colored chips may be removed.

4.2.4 Arguments in Favor of the Neutralization Model

The motivation of these authors for recommending the neutralization model is as follows:

[1] The inadequacy of the number line, which is "abstract" (Dirks, 1984), "incomplete" (Battista, 1983) and "confusing (Grady, 1978), and uses "mystifying manipulations" (Haner, 1947). The number line gives "trouble" for the operation of adding a negative number (Bartolini, 1976), and causes confusion between the notions of shifts and labels (Baily et al., 1974).

[2] The neutralization model is a <u>concrete</u> model, representing an abstract idea (Grady, 1978), and takes a concrete approach to operations (Bennett & Musser, 1976). While the number line is pictorial, this model is concrete, which students need in initial instruction (Battista, 1983). The concrete operations give meaning to arithmetic statements (Dirks, 1984). The concrete nature of the model is good for remedial students who can see the illustration of general rules (Whimbey & Lochhead, 1981).

[3] There is not a long-term dependency on the physical manipulations, and the model quickly <u>becomes a mental model</u> (Cotter, 1969), which alleviates the need for abstract rules (Jencks & Peck, 1977), but gives an opportunity for their development (Whimbey & Lochhead, 1981). Children are able to manipulate mental objects (integers) by composing and decomposing them in the same way that they did so with the physical objects.

[4] Operations on integers should <u>correspond as much as possible with the known system of natural numbers</u> (Tobias et al, 1982), and this model uses the notions of combining for addition and take-away for subtraction (Fremont, 1966) which is an extension of whole number operations (Bennett & Musser, 1976), behaving like the model used for whole number arithmetic (Gibbs, 1977), where addition and subtraction are inverse operations (Semadeni, 1984). This take-away procedure for subtraction "gives a correct intuition for $^{-}9 - ^{-}2 = ^{-}7$" (Bell, 1983). This model gives equal status to positive and negative numbers (Freudenthal, 1983).

[5] The neutralization model is <u>consistent with notions in other fields</u>. The charged particle concept "does not violate scientific evidence", and provides the foundation for the concept of ion exchange (Cotter, 1969). The hot and cold cube approach uses the child's perception of temperature changes (Jencks & Peck, 1977). The use of ordered pairs is consistent with the logical structure of modern mathematics (Sherzer, 1969).

[6] This model provides a <u>physical representation of mathematical generalizations</u>, such as: additive inverse (Cotter, 1969, Battista, 1983); subtraction in terms of addition (Cotter, 1969, Gibbs, 1977); commutativity, associativity and additive and multiplicative

identities, (Battista, 1983). It eliminates the confusion over opposites and negatives, and over subtraction and negatives (Gibbs, 1977). Representing integers by colored counters is close to the theoretical mathematical approach of using ordered pairs to represent integers (Semadeni, 1984).

[7] The ease of presentation of this model means that it needs no preliminary instruction, and is straightforward and simple (Dirks, 1984), is easily visualized and can be suitably presented (Cotter 1969, Haner 1947).

[8] Conceptual learning is stimulated by simultaneous manipulation of concrete material and integer number sentences (Whimbey & Lochhead, 1981), and the physical operations will help students to anticipate results of integer operations (Dirks, 1984).

[9] The model not only works for addition and subtraction, but according to Gibbs (1977) it can be used for multiplication. For Battista (1983), it allows for teaching of all four operations in a meaningful way. This argument neglects the fact that multiplication and division are not integer by integer operations using this model.

[10] Experience has proven the model to be effective. Students who were taught this model in a remedial way had better scores on post-tests (Whimbey & Lochhead, 1981). It has been very successful in giving elementary school teachers a feel for integer arithmetic, and students quickly pick up these simple operations (Gibbs, 1977). The model (in the form of hot and cold cubes) has been "so successful" for integer addition and subtraction that it "must" be passed on (Jencks & Peck, 1977).

[11] It uses a discovery process of learning (Sherzer, 1969, Fremont, 1966), and should promote observation and discussion (Haner, 1947).

[12] It will help children learn how to add a negative number (Bartolini, 1976), and will give them experience with the concept of negative numbers (Tobias et al, 1982).

[13] It will give meaning to integer operations in such a simple and direct way that rules will no longer be used without thinking about the situation - for example, (+5) - (+3) will not be automatically

performed by a rule, but by reasoning about the situation (Fremont, 1966). It provides a clearer understanding of the subtraction process, and avoids premature abstraction (Haner, 1947).

4.2.5 Concerns About the Use of the Neutralization Model

Some of these authors express concern about the use of the annihilation model. These are as follows:

[1] Fremont (1966) feels that the drawback to using this model is that it cannot be used for multiplication and division.

[2] Kohn (1978) stresses that students must be involved themselves in the physical activity.

[3] Dirks (1984) says that it is essential to perform both the concrete and arithmetic operations simultaneously, so that children will begin to anticipate results of operations on integers.

[4] Battista (1983) does not recommend using this model in isolation, but as one of several models.

Other authors who have expressed concern about the use of this model for teaching integers are:

[1] Fischbein (1977), who feels that it lacks power and becomes a "blind alley" since one is not able to use it for rational and real numbers. He expects that one strength of a model is that once a mathematical situation is translated into the terms of the model, the problem should be able to be solved within the model, and since the chips are countable within the whole numbers only, this does not suit extension into other number systems. The opinion of this author is that generalizations made from the model apply no matter what system one is working in. It is possible to solve rational problems in both fractional and decimal form, such as ${}^{-}4.5 - {}^{+}2.73$, even though they are not whole numbers, by the mathematization processes described above, and developed directly from the model (thus ${}^{-}4.5 - {}^{+}2.73 = {}^{-}4.5 + [{}^{-}2.73 + {}^{+}2.73] - {}^{-}2.73 = {}^{-}4.5 + {}^{-}2.73 + [{}^{+}2.73 - {}^{+}2.73] = {}^{-}7.23$).

[2] Bélanger (1984), who states that the notion of equivalence classes, necessary for subtraction, "est particulièrement difficile à saisir pour les élèves du primaire et meme du début secondaire". He

also states that in his experience of experimenting with this "mathematical" model and with the "intuitive" temperature model, only gifted pupils do not have difficulties with subtraction using this notion of equivalence classes.

[3] Galbraith (1974) states that the procedure of using an equivalent representation for the minuend in a subtraction task (in particular, with the ordered pairs representation), is one that requires a student to operate at Piaget's Formal Operational stage, which she designates as happening not before the age of 13 or 14.

The weaknesses that can be seen in the articles advocating the annihilation model are as follows:

[1] Those authors who do attempt to make this model work for multiplication (Battista 1983, Kohn 1978, Dirks 1984, Cotter 1969) interpret the leading sign as one of repeated addition or of repeated subtraction, and treat the leading "number" part as a whole number, rather than truly performing the operations on two integers.

[2] No one author deals explicitly with all of the notions inherent in this approach to integer operations (opposites, neutralization, neutral element, equivalence class of zero, equivalent representations, mathematization, etc.)

[3] The approach that Tobias et al. (1982) take by first introducing a procedure called "Cross Tots" is very abstract and convoluted, compared to the simplicity of the model itself. Indeed, even the hot and cold cubes (Jencks & Peck, 1977), the holes and plugs (Gibbs, 1977) and the pipe cleaners (Fremont, 1966) seem like contrived situations compared to the straightforward approach of positive and negative chips or symbols.

[4] It appears that none of these authors who attempt to go beyond the physical manipulations are willing to see subtraction remaining with a "take away" meaning, a very nice feature of this model. They all seem concerned that the student be led to the rule of "add the opposite" in order to formally subtract. They rarely mention that this equivalence of operations can be used to full advantage in an additive situation as well (i.e. subtract the opposite), and therefore miss out on the full notion of equivalence. It also seems to be a manifestation of "avoidance" of subtraction.

[5] While the abacus method (Dirks, 1984) seems to work nicely, it is based on a split number line, which has been an historical obstacle to the understanding of integers, evidence of which will be presented in Chapter Three.
[6] Very few authors use the model to its full potential, i.e. in mathematization. They expect the normal rule formation, which separates itself from even a mental model to refer to. Only Bennett & Musser (1976) and Cotter (1969) use the decomposition of integers, and Tobias et al. (1982), Baily (1974) and Sherzer (1969) use ordered pairs to mathematize the physical operations of adding to and taking away from.

4.2.6 Comparison of the Number Line and Neutralization Models

Using the criteria developed in Chapter Three for understanding of integer notions within the context of a number system, and considering the features of the two models explained in detail in this chapter, a comparison of these models may be made in order to determine how well each model is suited to be used for the construction of integer notions and the operations of integer addition and subtraction. Table 3 shows how the two models demonstrate integer notions.

Table 3: Comparison of Integer Models

NOTION	NUMBER LINE MODEL	NEUTRALIZATION MODEL
opposites (inverse element) $^-6 + ^+6 =$ $^+6 + ^-6 = 0$	$^-6$ and $^+6$ are opposites since (a) they represent the same distance, but in opposite directions (b) they represent positions symmetrical about zero, opposite to each other (c) addition of their displacements results in a return to the original position	$^-6$ and $^+6$ are opposites since they are equal amounts of opposite kinds, and thus unite to form a neutral element.

zero (identity element) $n + 0 = 0 + n = n$	zero is (a) origin (position), (b) no displacement, and (c) represents no movement	zero is (a) neutral, and (b) "nothing"
concrete embodiment of integer notation ($^-4$, $^+9$)	- indicates position (magnitude and direction) of an isolated integer (declarative) - raised sign sometimes indicates direction of displacement of number in an operation (i.e. takes on an operational meaning)	- indicates quantity (how many) and kind (positive or negative) - raised sign directly linked to type of chip (positive chip or negative chip) and is therefore declarative
equivalence classes: $(m, n) = (m+y, n+y)$		any integer, including zero, may be represented with an infinite number of combinations of positive and negative chips
ordering	by position (more positive)	by quantity (absolute value)
closure under subtraction	all subtractions possible, but not all easy to model (some authors avoid this notion by changing subtraction to addition	all subtractions possible, but some require an intermediary step
procedures for addition and subtraction		
[5] outcomes of operations	focus on movement and not on reasoning	focus on reasoning, more intuitive

From the above, it can be seen that there are inadequacies in both models. With the number line, the notions are bound to the model, and do not exist outside of the model as number notions. With the neutralization model, there is no notion of order based on direction, and therefore no discrimination between the sizes of $^-5$ and $^+5$. Zero is not an origin from which one can move in either direction, which could lead to the notion of the divided number line.

In addition, each model contains features (non-procedural) which could be potential obstacles to integer understanding. The number line represents number both by vector and by point, while the neutralization model's many equivalent ways to represent any integer may prove to be confusing (just as equivalent fractions are not always easily understood).

It can be seen that the strength of the neutralization model lies in its embodiment of integer notions and in its procedures for the integer operations of addition and subtraction. However, no distinctions for ordering exist within the model, and thus the model cannot stand alone, but must be used in combination with a model such as the number line to complete ordering notions and procedures (which will include notions about zero as origin).

The neutralization model appears to be a better model than the number line for addition and subtraction of integers due to its more concrete nature, its use of operation strategies that are known in N, and its ease of presentation. It remains to be seen whether in a teaching situation the addition of the neutral element before subtraction proves to be such a natural, reasonable thing to do, or if it is too much of an abstraction even though it somewhat parallels "borrowing" in N and Q. The weakness of the model is that it does not seem to work "naturally" for integer multiplication and division, and it does not deal with conventional ordering (where the number line is of benefit).

4.2.7 Use of the Neutralization Model in Educational Documents

Although an overwhelming percentage of teachers and textbooks advocate the number line model, some educational authorities are beginning to turn to the neutralization model. Unfortunately, documents produced have shown an incomplete version of the model, and indeed a lack of understanding of the key concepts and advantages this model provides.

(a) Curriculum Documents

According to Janvier (1985) this model is included in the national curriculum in Morocco, but no attempt was made to obtain this material.

Four documents which advocate the use of a neutralization model to teach integer concepts were analyzed in terms of their approach to the notions of the model. The findings are as follows:

[1] Britain (Computation and Structure, 1969): The Nuffield Guides emphasize "how to learn, not what to teach". Integers are introduced as inventions for solutions to equations such as $5 = 7 + \square$, where (5,7) can be considered the ordered pair that specifies the problem, and the "solution" is $^{-}2$. Since there are other number sentences, and hence other pairs that may have $^{-}2$ as the solution, the notion of equivalence class is introduced, and much practice in making equivalent ordered pairs using different colored counters is encouraged. Addition is dealt with as just another representation of an ordered pair, and rather than matching opposites, one just counts all to make the ordered pair, then finds the lowest representation (with no method given for this operation). Sums do not depend on the particular ordered pairs used to represent them (i.e. there are many ways of illustrating that $^{+}6 + {}^{+}1 = {}^{+}7$). Zero results from combinations of equal numbers of different colored counters, and is the "neutral element" since adding zero to any number does not change the number. Subtraction is dealt with as finding the missing addend, and instead of using counters, one is to use numbers (for example, $^{+}3 - {}^{-}4 = \square$ becomes $^{-}4 + \square = {}^{+}3$ so adding $^{+}4$ to both sides yields $^{+}4 + [{}^{-}4 + \square] = {}^{+}4 + {}^{+}3$, so $[{}^{+}4 + {}^{-}4] + \square = {}^{+}7$, so $\square = {}^{+}7$, and therefore $^{+}3 - {}^{-}4 = {}^{+}7$), and thus one can see that subtracting a negative is like adding a positive.

The obvious criticism is with the approach to subtraction. Not only does this guide not use the notion of take-away provided by the model, but it uses techniques which fall into the category of "equation-solving" (for example, adding $^{+}4$ to both sides), which may not be mastered at the time that integers are taught. The action of combining sets is not reserved exclusively or explicitly for addition, and the resultant value notion seems blended in with the idea of addition. The concepts of opposites and neutralization are not a part of this description of the model. The student who performs addition has a concrete model to use, but for subtraction, there is no such support.

[2] Quebec (Relative Integers, 1986): This curriculum guide is designed to provide an introduction to integers for students in grades 5 and 6, where the topic is an optional one. At first glance, it appears that this guide is a duplication of the above Nuffield guide, as even the diagrams correspond to the British document. However, some new features appear. Resultant values are made on the criteria of "dominance" (the term used by French descriptions of the model), and addition is performed on groups of counters. Although most of the document contains negative integers with the proper notation, before subtraction is introduced, negative integers are written in red (in the present text, an italicized outlined number will be used to correspond to the different color), and statements like $4 + \mathit{4} = 0$ are made, where $\mathit{4}$ is called the "additive inverse" of 4. Addition is then mathematized by decomposition as $13 + \mathit{4} = (8 + 5) + \mathit{4} = 8 + (5 + \mathit{4}) = 8 + 0 = 8$. The procedure for subtraction initially uses finding the missing addend for ordered pairs, where it may be necessary to change the representation of the initial ordered pair.

[3] Manitoba (Integers and Combining Like Terms in Algebra, 1986): This document, which was designed to avoid the confusion of the double signs, misuse of language, and an abstract approach, uses unmarked blue and red bingo chips, and finds resultant values in terms of red or blue: (Of which color are there more? How many more? 2 blue and 4 red = 2 red). It is given that one blue and one red make zero, and therefore they are opposites. Later, plus and minus signs are put on the chips, and the same questions are asked, with the answer expressed as "2 negative and 4 positive = 2 positive".

The emphasis is on the word "and", and at no time is there ever an attempt to explicitly add. At the next stage, and this differs from the conventional description of the model, each chip is marked with a sign and a numeral, for example +5. Resultant value of a group such as +3, -4, +1 and -5 is determined by the same two questions (which kind has more? how many more?), and the statement is written as $+3 - 4 + 1 - 5 = -5$, yet the statement is still read as "positive 3 and negative 4 ...", and not as an addition sentence, although the sign which was intended to designate positive or negative appears visually to designate an operation.

Next, a negative sign before a number is defined as indicating the opposite of that number, so that -6 is the opposite of 6, and -(+3) is -3 (i.e. the opposite of +3 is -3). Addition (including subtraction) is now performed not on concrete materials, but on number strings such as $+3 - (+2) - (-4)$ which is then translated by the notion of oppositeness to $+3 - 2 + 4$, and still read as and rather than as addition. This document then leads into combining like terms in algebra.

This document provides a very disappointing interpretation of the neutralization model, and clearly has algebraic gathering of like terms as the goal of instruction, rather than understanding of integer operations. Thus a formal approach is adopted almost from the beginning, and the operations of addition and subtraction are grouped into an "additive" procedure based on dominance, designated by the term "and".

[4] Boston (Integers and Computation with Integers, 1991): This booklet was designed as a textbook for adults taking high school mathematics in university. Equivalent representations of integers using colored chips, and neutralization are explicitly explained. Addition and subtraction are performed exactly as outlined in the description of the neutralization model, and there is no mathematization, except for the generalization of "how many would zero out?" when larger numbers are encountered.

(b) Textbooks

Of ten fairly recent grade 7 mathematics textbooks analyzed (Appendix B), only two used the neutralization model exclusively to teach operations, while two others used it in combination with the number line. Although at the high school level, students do not really read textbook arguments, teachers do not always have access to instructor's manuals, and thus tend to teach from the textbook. The explanations given in the texts which rely exclusively on this model are as follows:

[1] Mathquest 7: Integers are represented as red and black squares (black = you have \$1, red = you owe \$1) put into a "bank" where you must determine the "balance" (resultant value). The students are asked to make their own squares and to manipulate with them. After determining the balance of like colors, addition is introduced by asking the student to add squares to the bank and to write a number sentence describing the action. However, no examples are given which show answers, and no explanation is given about neutralization, so the procedure for additions of unlike signs is missing.

Next, the equivalence class of resultant values is introduced, by asking the student to put in or take out equal numbers of red and black squares, and then to determine the balance. Only lower on the page is an actual value given as a balance, and then the student is asked to show different ways to make particular integers.

Subtraction when there are not enough to take away is treated in a vague manner. For example: "Subtract $^{+}3$ - $^{-}2$. To take 2 red counters out of the bank, there must be 2 red counters in it. Hence, we show +3 using 2 red counters and *some* black counters." (italics this author's). Other examples are given non-verbally, and then a comparison between taking black counters out or putting red counters in is made, and the statement, "This suggests that, to subtract an integer, we can add its opposite." is given.

There is no mathematization, and no large number tasks are given.

[2] Réalités Mathématiques 1: Pictures of colored dots are used, and the student is asked to imagine that they represent money received or spent, or points gained or lost in a game. Resultant value of a group is determined implicitly by annihilation, as a dot of one color with a dot of another color makes both dots "disappear". These dots are then called negative and positive, and addition sentences are written to document various combinations, using "annihilation" of equal amounts of opposite colors. Then decomposition is shown, and is to be used only if necessary, and examples using large numbers are given.

Subtraction is introduced in an abstract fashion, where complete subtraction sentences are given, and the student has to write the corresponding "missing addend" sentence, then solve it, and verify that the subtraction sentence given is correct. Following this, in isolation, the rule "pour soustraire un nombre on ajoute son opposé" is given, followed by subtraction exercises. It is amazing that a text which introduces integer addition so well using the neutralization model not only abandons the model for subtraction, but also does such a poor job of presenting subtraction.

4.2.8 Reported Research on the Neutralization Model

Although many published articles, indicated above, advocate the use of the neutralization model to teach integer operations, very few report any research which tested the model's success, and none have investigated the weaknesses or obstacles which may be inherent in the model. As well, no one has attempted to systematically analyze the mathematical notions which are integral to the use of the model. The following teaching experiments were performed:

[1] Liebeck (1990), motivated by Kuchemann's (1981) recommendation to regard integers as discrete entities and to abandon the number line, used two groups of grade 3 and 4 students (10 in each group) to compare instruction on the integer operations

of addition and subtraction, one group using the "scores and forfeits" interpretation of the neutralization model, and the other using number line movements where operations with a negative meant first facing in the negative direction. The lesson content was as follows:

	SCORES AND FORFEITS	NUMBER LINE
1	- game: notions of resultant value, equivalence of representations, equivalence of operations - writing of word descriptions, inventing notation for "forfeits"; inventing notation for zero	- extend number line to left; notation - addition and subtraction of whole numbers from any position, physically performed on playground number line and on worksheet - writing of sentences
2	- game: writing of number sentences using resultant values	- addition and subtraction of negative numbers on playground number line and on worksheet - writing of sentences
3	- completing sentences "as if each recorded some isolated turn in a game" - change representation for subtraction - worksheet (16 items)	- review - worksheet (16 items)

Six weeks after instruction, both groups were given a 10 item test, with all numbers between $^{-}3$ and 3 (for the scores and forfeits group, negatives were written as whole numbers circled in red). The last 4 items involved a string of 2 operations (e.g. $2 + {}^{-}1 + {}^{-}3$), and had not been encountered in the instruction, except by some of the first group. For the first six items, the scores and forfeit group scored slightly higher than the number line group on all items, and both groups scored lower on subtractions, especially when the signs were different.

[2] Rowland (1982), motivated by the lack of success in subtraction tasks of middle and low ability pupils using the number line, conducted a teaching experiment, which he describes as an "initial feasibility trial", of four half-hour lessons with a group of four 11 year olds. These lessons were based on the neutralization model, in particular on the notion of "equivalent representations", and were structured in the following way:

<u>Lesson One</u>: (using black and red cubes) neutralization; opposites; neutral pairs; equivalence of piles of cubes with respect to removing or adding neutral pairs; notation $\binom{n}{m}$; and abstraction of equivalence to 3 digit number pairs.

<u>Lesson Two</u>: review of equivalence; addition of piles of cubes; recording of results as $\binom{3}{1} + \binom{2}{5} = \binom{5}{6}$; addition of equivalent piles which result in equivalent answers; addition of number pairs.

<u>Lesson Three</u>: use of a thermometer to introduce conventional integer notation (raised sign with numeral); identification of black cubes as positive number, red cubes as negative number; mathematical notation as representing an equivalence class of piles of chips (for example $^{+}2 = \bullet\ \bullet = \binom{2}{0} = \binom{3}{1} = \binom{4}{2}$ etc.); addition using "smallest piles" (canonical representation), first with cubes, then with integer notation in format of $^{+}2 +$ $^{-}3 = \binom{2}{0} + \binom{0}{3} = \binom{2}{3} = \binom{0}{1} = {}^{-}1$ to record how the cubes were used; preliminary discussion on subtraction.

<u>Lesson Four</u>: review of equivalence, addition and trivial subtraction of integers; non-trivial subtraction with cubes and integer notation.

The students had no difficulty with additions of cubes, but although Rowland was motivated to try this model particularly to gain success with subtraction, responses to lesson four were not as intuitive as those previously taught, and the subtraction procedures were not as well developed by the students as the earlier procedures were. No post-testing was performed. Rowland states that it would be his goal to use this model to lead to the procedure of "adding the additive inverse", which he sees as the "only viable definition" of formal subtraction.

[3] A series of projects submitted as a partial fulfillment for the MTM program at Concordia University describe small scale (usually one student) teaching experiments using the neutralization model. These experiments were remedial in nature with one exception (Goncalves 1982), and dealt with weak students (Nabbie 1979, Chalouh - Hertzman 1979, Luckow 1982, Rabin 1979). In addition, Brien (1979) attempted to teach a class of 14 weak grade 8 students who were 2 years behind in their schooling. Student success was measured in terms of performance on items of integer addition and subtraction, and most individual remedial students achieved success, whether the model was in the form of holes and plugs (Nabbie, Chalouh - Hertzman), positive and negative polarities (Brien, Luckow) or colored chips (Goncalves, Rabin). No attempt was made to mathematize from the model, and two-thirds of the in-service teachers expected that the model would justify the subtraction rule, but reported that this was not the case. All of the individuals were described as weak students, and difficulties were reported with (a) an overriding response to try to use ill-remembered rules (grade 10 student); (b) an inability to abstract from cardboard holes and plugs to paper and pencil tasks with symbols (grade 9 student); (c) difficulty in generating rules from the model (grade 10 student doing grade 9 math) (d) an inability to see connection between model and subtraction "rule" (grade 7 student). Brien's experiment to design a series of lessons for an entire class did not have as much success, and took longer than she had anticipated.

[4] A teaching experiment using colored chips was also designed and executed by Lytle and Avraam (1991) to remediate individually with a grade 9 and a grade 10 student who had each demonstrated (and verbally expressed) difficulty with integer subtraction, but who were moderately successful with integer addition. The students were introduced to the model as an alternate one for additions, and comparisons were made between addition results using the students' strategies and results obtained from manipulation with colored chips in order to give the students confidence in the model. Then integer subtraction was introduced using equivalent representations and "take away" subtraction. The experiment was successful in that the students were not only able to solve operations with simple integers, but also displayed the use of a mental model based on the concrete model for any type of integer addition and subtraction, when the integers were 2 digit or larger. The students also expressed gratitude for being shown a method that they could rely on and that provided meaning for them.

4.2.9 Implications

The literature indicates that mose interpretations of number line operations may be inadequate to meaningfully model integer subtraction. The neutralization model appears to be advantageous over the number line model in its modelling of addition and subtraction as combining and taking away, and should provide a strong concrete basis for the students to construct their own mental model of integers. However, it appears that this model is not often employed to its full potential in this regard. It then remains to do an analysis of integer notions, and integer addition and subtraction based on the neutralization model, and to use this as a basis for the design of a teaching experiment. The goal of the instruction would not be to remediate, but to introduce the concept of integers and the operations of integer addition and subtraction within the environment of the physical model, and to bring these notions to the stage of mathematization (use of a mental model for integers) from the concrete model.

CHAPTER FIVE

ANALYSIS OF INTEGERS AND INTEGER ADDITION AND SUBTRACTION

5.0 ANALYSIS OF UNDERSTANDING OF INTEGERS

Although many authors have written about the procedural use of specific models which may be used in the learning of integers, there is very little information about the nature of integer understanding contributed by the particular model. One author has made an analysis of levels of integer knowledge within two physical models (but has not extended this to integers as a number system).

The present author's analysis in this chapter seeks to define what it means to understand integers both within the context of the concrete neutralization model, and within the domain of integers as a number system. This analysis contains procedural knowledge as well as understanding of integer concepts.

5.1 ANALYSIS OF INTEGER KNOWLEDGE

Peled (1991) has analyzed the levels of "knowledge" leading to understanding addition and subtraction of integers from the perspectives of both a number line and a "quantity representation". What he provides is a list of mostly performance items, bound to the physical model for the former (when to move right or left), and to intuitive notions about quantity for the latter. These levels, intended to be in a linear progression with the exception of the first two levels in the quantity dimension, are given in Table 4 below.

Table 4: Levels of Knowledge about Integer Operations

LEVEL	NUMBER LINE DIMENSION	QUANTITY DIMENSION
1	Notion: existence of negative numbers to the left of zero, symmetric to the natural numbers Procedure: order is determined by the rule: the larger number is on the right	Notion: existence of negative numbers, which are amounts of things (of an unfavorable characteristic) Procedure: order is defined in an "inverted way" to the natural numbers: i.e. the larger the amount of negatives, the smaller is the number, since it is a worse-off state.
2	Procedure: (a) subtracting below the zero for whole numbers (e.g. 2 - 8) (b) starting at a negative number, add a positive number (although he doesn't mention it, presumably subtraction of a positive number from a negative is also at this level)	Procedure: ability to extend the operation of subtraction to allow for cases such as 5 - 9 by taking away the available amount, then "figuring out" the amount missing, and giving it a negative label. Notion: notation for negative numbers
3	Notion: distinction between the "positive" and "negative" worlds Procedure: (a) addition within negatives (neg + neg) means making more negative, so move to the left (b) subtraction within negatives (neg - neg) means becoming less negative, so move to the right.	Procedure: willingness to extend notions of addition and subtraction of whole numbers to addition and subtraction of amounts of the same kind, with subtraction of negatives restricted to the cases where there are enough available to take away.

4	Procedure: abstract from the above that to add a negative, move left, to subtract a negative, move right.	Procedure: addition and subtraction of quantities of different types, where the effect of the operation is determined by both the operation and the sign of the second number. This requires the mental ability to make decisions on whether a "better" (add a natural number) or "worse" (subtract a natural number) situation should result.

The levels given for the number line dimension may be interpreted as follows:

Level 1: Extension of number line knowledge to include existence of negative numbers.

Level 2: Preservation of procedures for operations in N, but now (a) cross over zero, and (b) begin to operate from the other side of the number line.

Tasks: $^{+}S - {}^{+}L$, $^{-}S + {}^{+}L$, $^{-}L + {}^{+}S$, $^{-}S - {}^{+}L$, $^{-}L - {}^{+}S$

Level 3: Distinction made between areas of positives and negatives, use of opposite operational strategies

Tasks: $^{-}S + {}^{-}L$, $^{-}L + {}^{-}S$, $^{-}L - {}^{-}S$, $^{-}S - {}^{-}L$

Level 4: Generalization of rules to allow for awkward (all other) cases.

Tasks: $^{+}L + {}^{-}S$, $^{+}S + {}^{-}L$, $^{+}L - {}^{-}S$, $^{+}S - {}^{-}L$

It is evident that the justifications used for level 3 use reasoning about a quantity notion of number, and not about geometrical number line notions. In fact, for number line knowledge, levels 3 and 4 should be grouped under use of a rule that states that if the number following the operation sign is negative, go the opposite direction from which you would normally go (add a negative means go left, subtract a negative means go right). He does not specifically deal with the addition of opposites.

The levels given for the quantity dimension can be thought of as follows:

Level 1: Quantity notion of negative numbers.

Level 2: Subtraction of two natural numbers by decomposition (how many are left over when the match is made). This could identify the subtraction sign with negative numbers, since these subtractions end up with a negative answer.

Task: $^{+}S - {}^{+}L$

Level 3: Extension of operations procedures from N

Tasks: $^{-}S + {}^{-}L$, $^{-}L + {}^{-}S$, $^{-}L - {}^{-}S$

Level 4: Abstractions about positive and negative numbers.

Tasks: $^{+}L + {}^{-}S$, $^{+}S + {}^{-}L$, $^{-}L + {}^{+}S$, $^{-}S + {}^{+}L$, $^{+}L - {}^{-}S$, $^{+}S - {}^{-}L$, $^{-}L - {}^{+}S$, $^{-}S - {}^{+}L$

The reference in level one to negatives as amounts of "unfavorable" character has two shortcomings. The first is the affective association of negative number with a negative attitude, and the second is that negative numbers do not necessarily always represent "bad" things, and thus this should not be a criteria for understanding of integers.

There are some problems also with his progression through the levels. The tasks at level 3 are easier than the task at level 2, since once integers are given a quantity representation, these tasks represent possible additions and subtraction of quantities, negative in nature, while the subtraction in level 2 is of a much more subtle nature. There is a major jump between these levels and level 4. With limited exposure (only 4 types of tasks), the students are expected to reason about quantities, and predict the nature of outcomes. Although the procedure given for operations of integers of different signs (nowhere does he tackle $^{-}S - {}^{-}L$) is to first determine the nature of the outcome, and then to change the problem to one involving a natural number as the number following the operation sign, applying this rule to half of the tasks at this level <u>still</u> yields an operation on integers with different signs. The suggested reasoning changes the subtraction of a negative from a positive into the addition of positives, but the subtraction of a positive from a negative (giving a worse condition), changes to the subtraction of a natural number, leaving the task unchanged, with no procedure to follow. The same situation occurs when a positive is added to a negative (better condition, add a natural number).

Peled reports that twenty grade 6 students who had received integer instruction in grade 5 had been interviewed once to evaluate their level of knowledge based on the above designations, and although he states that the analysis was incomplete, he expects that most of the twenty would be at level one or two for number line representations and within levels one to three for the quantity dimension.

He has not used the notions of opposites, neutralization, equivalence classes or mathematization as criteria, which are inherent to the neutralization model's approach to "quantity dimension". Thus his levels appear to be limited to levels of procedural knowledge within a given model rather than being a complete analysis of levels of understanding of integers. This raises the more generic question "What does it mean to understand integers and their operations?".

5.2 UNDERSTANDING OF INTEGERS USING THE NEUTRALIZATION MODEL

The following is an analysis of the notions and procedures that will be used to design a teaching experiment that will focus on the understanding of integers using the neutralization model as the main tool and the number line to supplement it when necessary. The order of presentation is linear in terms of presentation of notions, but spiral in terms of the building of each notion. These notions are divided into five sections, each section comprising a teaching unit. Some of these notions concern the understanding of integers as numbers, and others involve procedures on integers. Most begin with building understanding first within the context of the concrete model using colored chips, then extend this understanding to integers as mental (mathematical) objects.

5.2.1 UNIT ONE General Notions About Integers (introductory)

(1) OPPOSITENESS

Integers are numbers which have opposites, and at the physical level, this quality of "oppositeness" is an intuitive one. There are many examples in everyday life of opposites in operations (rise - fall, expand - shrink), in qualities (hot - cold, good - bad), and in entities (day - night, top - bottom). At the intuitive level, opposite "charges" may be introduced as different colored poker chips, labelled + or -, and called "positive" and "negative" respectively. These chips may be symbolically represented on paper with a + or - sign, and will be thus represented from this point on in this analysis.

(2) INTEGER NOTATION FOR UNIT CHIPS

The integer notation of $^{+}1$ for one positive chip and $^{-}1$ for one negative chip will be used to represent the value of a unit chip. It is hoped that the emphasis of this model on positives and negatives would minimize any possible confusion over the sign of the number and the sign of the operation. At this stage, the raised positive sign for positive integers (which texts often drop) will be retained to avoid confusion with the natural numbers.

Another common notation used is that of bracketing integers, or only negative integers, for example (-1), and (-1) + 3, or (-1) + (+3). When integer operations are presented in a purely mathematized environment, it is thought that the brackets would become too cumbersome, for example (-6) + (+2) = [(-4) + (-2)] + (+2). Therefore this notation will not be used here, except as an alternative if the raised sign notation proves to be difficult for any student.

(3) RESULTANT VALUE OF GROUPS OF CHIPS OF LIKE SIGN

When a collection of chips contains exclusively positive or negative chips, one may simply count the number of chips in the collection to find the resultant value. For example, - - - - has a resultant value of 4 negatives. This procedure is compatible with the determination of quantity in the set of natural numbers.

(4) SYMBOLIC NOTATION OF RESULTANT VALUE OF CHIPS OF LIKE SIGN

When each chip in a grouping is of the same sign, the standard integer notation, of for example $^{-}4$ to represent 4 negative chips or $^{+}7$ to represent 7 positive chips, should present no difficulty.

(5) NEUTRALIZATION OF ONE POSITIVE AND ONE NEGATIVE CHIP

Since a positive chip is opposite to a negative chip, when one is paired with the other, the effect is one of neutralization, and the result is neither positive nor negative. The student may have experienced the idea of neutralization within a concrete life-based framework, even though he may not have yet encountered the science-based neutralization of acids and bases or the atomic theory of charged particles within an instructional setting. The situations of going up and down a staircase and the opening and closing of a door are specific examples of neutralization of an action within the student's experience. A neutral condition (not hot, not cold) or position (not left or right of a designated position) should also be part of their life experience.

(6) CREATION OF THE NEUTRAL ELEMENT FOR CHIPS

(a) While many opposites which cancel each other return the situation to a state of neutrality (for example, moving forward one step, then backwards one step returns one to the initial position), the cancellation of equal numbers of positives and negatives <u>creates a neutral element</u>. The term **"neutral element"** will be used here and throughout in the following way: a combination of one negative chip and one positive chip, $\boxed{-\ +}$, is a neutral element, with no effect or quality of positiveness or negativeness. The designation for this combination of one chip of each kind will be "unit" neutral element. A combination of equal numbers of positive and negative chips, for example $\boxed{\begin{matrix} - - - \\ + + + \end{matrix}}$, is also defined as a neutral element since it has the same effect and quality as a unit neutral element. Note that when a negative chip is combined with a positive chip, each individual chip becomes a "neutralized" element.

(7) SYMBOLIC NOTATION FOR THE NEUTRAL ELEMENT FOR CHIPS

We will introduce the symbol NE as a kind of shorthand to represent this neutral element. The integer notation of 0 will be used after the procedure of addition of opposite integers, since the idea of zero as neutral does not exist for them at this point.

(8) NEUTRALIZATION PROCEDURE FOR CHIPS OF EQUAL QUANTITY

Just as one step forwards and one step backwards neutralize each other, so subsequent steps of equal magnitude will also bring one back to the initial position (i.e. have no effect on the initial position). In the same way, equal numbers of positive and negative chips will neutralize each other. The procedure for this neutralization is to match each positive chip with each negative chip to form neutral element units.

(9) RESULTING EQUIVALENCE CLASS OF THE NEUTRAL ELEMENT

Since the pairing of positives and negatives when there are equal numbers of each results in a number of neutral elements, the group of pairings may also be considered to be a neutral element, as stated above in (6). Thus there are an infinite number of neutral elements, all equivalent in that they are neutral and have neither positiveness nor negativeness.

Thus, for example, $\begin{matrix} + \\ - \end{matrix} \equiv \begin{matrix} - - - - \\ + + + + \end{matrix} \equiv \begin{matrix} + + + + + + + + + \\ - - - - - - - - - \end{matrix}$

(10) OPPOSITES OF INTEGERS

One positive chip was introduced as the opposite of one negative chip, and thus $^{+}1$ as the opposite of $^{-}1$. It has been shown that the neutral element may be created from or composed of equal numbers of chips of opposite signs, and one may infer here that the integers represented by these chips are also opposites. Thus, from the above diagrams in item (9), one can say that $^{-}4$ is the opposite of $^{+}4$ and that $^{+}9$ is the opposite of $^{-}9$ since the chips representing these integers neutralize each other. In general, each integer has an opposite, and that opposite has the same magnitude, but the opposite sign because each represents a group of the same number of chips, but of different signs. Thus the opposites of larger integers which cannot easily be represented by chips, such as $^{-}9748$, may be determined by mentally using the same criteria.

(11) RESULTANT VALUE OF MIXTURES OF POSITIVE AND NEGATIVE CHIPS

If both positives and negatives are present in a collection of chips, one must first rearrange the chips in a one - to - one correspondence, matching equal numbers of positives and negatives to form as many neutral elements (as much neutrality) as possible. The resultant value of the collection will be the value of the unmatched charges, as the neutral element by its very nature does not contribute to the value, and may be ignored. For example,

$$- + + + - + - + \text{ will become } \begin{array}{l} + + + + + \\ - - - \end{array}$$

which equals 2 positives. Thus the value of a collection of items is no longer "how many?" but has a method of quantification different from that in N.

(12) INTEGER NOTATION FOR RESULTANT VALUE OF CHIPS OF MIXED SIGNS

When the integer $^{+}2$ is used to represent the resultant value of the above grouping, it must be understood that this notation does not indicate the presence of the "extra neutralized chips" present in the physical model.

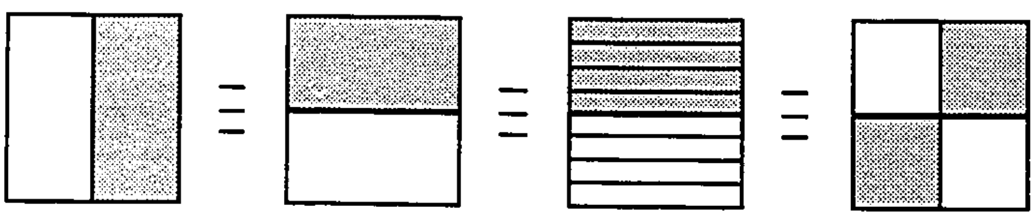
(13) EQUIVALENCE CLASSES OF POSITIVES AND NEGATIVES

Since the neutral element, in its infinite number of representations has no effect on the resultant value of a set of chips, any resultant value may also be represented in an infinite number of ways, by the presence of the neutral element in any of its forms. Thus

$$\begin{array}{ccccc} + + & \equiv & + + + + & \equiv & + + + + + + + + + + \\ & & - - & & - - - - - - - - \end{array}$$

are all equivalent representations of the resultant value two positives. The integer notation, +2, is now the canonical representation of the equivalence class of all these sets.

This notion should be familiar, as the concept of equivalent fractions is a similar one, and may be understood in a non-numerical way, as shown below by different representations of $\frac{1}{2}$.



5.2.2 UNIT TWO Integer Addition

(14) ADDITION OF CHIPS OF THE SAME SIGN

When the chips are of the same sign, one may combine them as in N, and "total" them by counting the resultant set. Thus - - plus - - - - becomes - - - - - - or 6 negatives.

(15) INTEGER NOTATION FOR ADDITION SENTENCE TO DOCUMENT ABOVE ADDITION

This is the first encounter with the combination of an operation sign and the sign of the integer. The above addition is written as $^-2 + ^-4 = ^-6$. The student must be able to discriminate between the addition sign and the negative sign for the 4.

(16) MATHEMATICAL ADDITION OF INTEGERS WITH LIKE SIGNS

This type of addition (for example $^+6 + ^+8$ or $^-2 + ^-10$) should be solved by adding the "number part" and noting the sign, and the neutralization model now becomes a mental model for this addition. This is similar in strategy to addition in N, and thus additions such as $^+945 + ^+3217$, which cannot easily be represented by chips, may now be solved.

(17) RULE-MAKING FOR ADDITIONS OF INTEGERS WITH LIKE SIGNS

The student of 12 years should be able to generalize the above addition procedure into some kind of formal "rule", which may occur spontaneously since it is expected that the model should induce it. An example of such a rule may be "When the signs of the integers are the same, add the numbers and keep the sign."

(18) ADDITION OF EQUAL AMOUNTS OF CHIPS OF OPPOSITE SIGN

When one adds equal amounts of chips of opposite sign, the result is the neutral element. Thus + + + + + + plus - - - - - - = NE.

(19) ADDITION OF OPPOSITE INTEGERS

When one adds <u>any</u> integer opposites, such as $^-10 + ^+10$, or $^+2345 + ^-2345$, it can be imagined that the result is neutralization.

(20) ZERO AS THE INTEGER REPRESENTATION OF THE NEUTRAL ELEMENT

The numeral 0 will be identified as the neutral element, with no positive or negative quality or sign, due to its neutrality. We can now complete the above number sentence as $^-10 + ^+10 = 0$. This relationship is necessary for the decomposition and application of the associative law for addition of integers of opposite sign.

(21) ADDITION OF CHIPS OF OPPOSITE SIGN

When quantities of positive and negative chips are added together, one must determine the value of the resulting grouping as described above under Resultant Value. The neutral elements are identified and isolated, and the remaining chips of a single charge determine the result. Thus + + + + + plus - - becomes

+ + + + +
- -

or 3 positives. Note that the procedure for matching to form neutral element(s) can either be a 1-1 matching or a group-group matching (3 positives neutralize 3 negatives).

(22) INTEGER NOTATION FOR ADDITIONS OF UNLIKE SIGNS

Just as the notation for the resultant value of a group of chips of unlike signs does not preserve the presence of the neutral element in the physical representation, so the addition statement does not indicate any neutral elements in the result of the addition. The statement for the above addition is ${}^{+}5 + {}^{-}2 = {}^{+}3$.

(23) ADDITION OF INTEGERS OF UNLIKE SIGNS

When the signs are different, the mathematical procedure for addition is one of decomposition in order to match groups of opposites and find the resultant values, as was done with the chips which were matched with a group-group procedure. The decomposition is not an arbitrary one, and the following two considerations must be made. The integer to be decomposed must be the one which represents the larger amount of chips, and the decomposition must be in a form where one of the two parts will be the opposite of the second number. For example, a decomposition statement would appear as follows:

$${}^{+}6 + {}^{-}5 = {}^{+}1 + {}^{+}5 + {}^{-}5 \text{ or } {}^{+}5 + {}^{+}1 + {}^{-}5.$$

The authors who explain this procedure use brackets to manipulate this string, as follows:

$${}^{+}6 + {}^{-}5 = ({}^{+}1 + {}^{+}5) + {}^{-}5 = {}^{+}1 + ({}^{+}5 + {}^{-}5).$$

In this experiment, for the sake of simplicity, the following cancellation procedure will be used, yielding a result of the second part of the decomposition.

$$^{+}6 + {}^{-}5 = {}^{+}1 + \cancel{^{+}5} + \cancel{^{-}5} = {}^{+}1$$

Note that these intermediate steps of decomposition and cancellation may be performed mentally.

(24) POSSIBLE RULE-MAKING FOR ADDITIONS OF UNLIKE SIGNS

It is possible that the student will notice that a short-cut to the decomposition process would be simply to subtract the numbers and keep the sign of the "larger" integer, but this is not necessarily the rule that may develop from using the neutralization model. It is more likely that the generalization may be termed as: "Match up part of the "larger" number with the "smaller" number, and see what's left of the "larger" kind". Again, this rule making should be induced from the model.

5.2.3 UNIT THREE Integer Subtraction

(25) SUBTRACTION OF CHIPS OF LIKE SIGN WHEN INITIAL GROUP CONTAINS ENOUGH TO REMOVE

The procedure for subtraction of positives and negatives is parallel to that in N, in that it involves removal of a subset of the initial set. "Trivial" subtraction presents no conflict, as it parallels subtraction in N, where the lesser amount may be "taken away" from the larger amount. Thus - - - minus - becomes - - , or 2 negatives.

(26) INTEGER NOTATION FOR DOCUMENTING SUBTRACTION OF CHIPS

As with addition, discrimination between the sign for subtraction and the sign of the second number must be made. The above subtraction is written as $^{-}3 - {}^{-}1 = {}^{-}2$, and documents the removal of one negative chip.

(27) MATHEMATICAL SUBTRACTION OF INTEGERS OF THE SAME SIGN WHEN ARITHMETIC SUBTRACTION IS POSSIBLE

The removal of chips now becomes a mental model for subtractions such as $^{+}371$ - $^{+}189$ or $^{-}207$ - $^{-}25$, where subtraction of the second number from the first is performed.

(28) RULE MAKING FOR SUBTRACTIONS OF INTEGERS FOR THE ABOVE CASE

The student should be able to generalize the procedure of subtracting the second number from the first, and keeping the sign for this kind of subtraction.

(29) SUBTRACTION OF CHIPS OF EQUAL VALUE

While addition of opposites results in the neutral element, subtraction of the same quantity results in "no chips left" - a true "annihilation". In one sense, this can be shown to be equivalent to the neutral element, since any resultant value may be represented in many ways, so that 5 positive chips minus 5 positive chips could be performed as follows:

+ + + + + + + +	minus	+ + + + +	=	+ + +
- - -				- - -

Although this technique of adding extra neutral elements to change the representation of the leading number for subtractions when there are not enough to "take away" will be used, it may seem rather artificial for this situation. Instead, the result of the subtraction of equal numbers of chips as "nothing" will be adequate.

(30) INTEGER NOTATION TO REPRESENT SUBTRACTION OF LIKE VALUES

When the above subtraction is documented, there is no integer symbol that has been introduced to represent nothing, so with integers, one can only write $^{+}5$ - $^{+}5$ = .

(31) IDENTIFICATION OF ZERO AS "NOTHING"

In the case of the above subtraction, when nothing is left after taking away, zero will be introduced as the integer that represents nothing. This is consistent with its use in operations in N, and one may now write ${}^{+}5 - {}^{+}5 = 0$.

(32) SUBTRACTION OF INTEGERS OF EQUAL VALUE

This is necessary for later "decomposition" of subtractions involving situations where there are not enough to take away. This now becomes trivial subtraction, paralleling the subtraction of equal values in N. We can now deal with subtractions such as ${}^{-}874 - {}^{-}874$. Note that the results of this subtraction are logical, and do not need contrived rules.

(33) SUBTRACTION OF CHIPS WHEN THERE ARE NOT ENOUGH TO REMOVE

When subtraction is physically impossible, as in 2 positives - 4 negatives (different sign), or in 3 positives - 5 positives (same sign, second amount larger), the response in N is that the question cannot be solved, that there is no answer. However, with the chips there is a procedure that will produce an answer. In either of these two cases, one must change the representation of the first set of chips by adding enough unit neutral elements (i.e. a combination of one negative and one positive chip) to provide enough chips of the type to be removed. This change of representation depends on the notion of Equivalence Classes of resultant values. Then the desired amount of chips may be taken away, and the resultant value determined by counting. For example, two positives minus four negatives would be modelled by putting out two positive chips, then adding four neutral elements, then removing four negatives.

$$+\;+ = \begin{matrix} +\;+\;+\;+ & +\;+ \\ -\;-\;-\;- & \end{matrix} = \begin{matrix} +\;+\;+\;+ & +\;+ \\ \boxed{-\;-\;-\;-} & \end{matrix} = 6 \text{ positives.}$$

Likewise, three positives minus five positives would be:

$$+ + + = \begin{matrix} + & + & + & + & + \\ & & & - & - \end{matrix} = \begin{matrix} \boxed{+ \; + \; + \; + \; +} \\ \hphantom{+ + +} - \; - \end{matrix} = 2 \text{ negatives.}$$

Note that when the signs are different, the number of neutral elements that must be added is equal to that of the number of chips to be subtracted. In the second case (when the signs are the same) this is not true, as less are required since some are already present, and the number needed may either be determined by counting on from the initial number, or by subtraction of the smaller amount from the larger amount, although if too many are added, the extra neutral elements have no effect on the evaluation of the result.

(34) INTEGER NOTATION FOR SUBTRACTIONS WHERE THERE ARE NOT ENOUGH TO REMOVE

The integer sentences which document the above subtractions do not reflect the addition of the neutral element to change the representation of the leading amount. The sentences reflect instead the initial quantities and the resultant value of the subtraction, as follows: ${}^{+}2 - {}^{-}4 = {}^{+}6$ and ${}^{+}3 - {}^{+}5 = {}^{-}2$.

(35) MATHEMATICAL SUBTRACTION OF INTEGERS WHEN NOT ENOUGH TO REMOVE

When subtraction is "impossible" because the first integer does not have enough to take away, the procedure is to re-write the problem by adding "zeros" to the minuend in the form of addition of opposites. When the integers have different signs, one simply has to add "zeros" in such a way that one member of the combination is the same as the subtrahend to allow cancellation, illustrated as follows (once again, while most authors use bracket manipulation to perform the procedure, this author will use cancellation):

$${}^{+}8 - {}^{-}3 = {}^{+}8 + {}^{+}3 + {}^{-}\cancel{3} - {}^{-}\cancel{3} = {}^{+}11$$

When both integers are of the same sign, one member of the combination of "zeros" must combine with the minuend so that subtraction of the subtrahend is possible. This component may be found by a missing addend approach ($^-4 + \square = {}^-9$). The integer subtraction may be illustrated as follows:

$$^-4 - {}^-9 = {}^-4 + {}^-5 + {}^+5 - {}^-9 = \cancel{^-9} + {}^+5 - \cancel{^-9} = {}^+5$$

Note that for the student, integer subtraction procedures may be mental processes rather than the above written procedures.

(36) POSSIBLE RULE MAKING FOR SUBTRACTIONS WHEN NOT ENOUGH TO REMOVE

At this point, the famous rule of "add the opposite" will not be a generalization, as no comparisons have been made about the results of operations. Since subtraction can be performed and understood without this rule, it is not an aim of this teaching experiment. Rather, the generalization that the larger number should have a new representation by adding zero to it may be the outcome of the use of this model. Thus a "rule" induced by this model might be "When subtraction is impossible, determine how many "neutrals" you must add until you <u>can</u> subtract, then determine the value of the remainder (using addition rules)."

5.2.4 UNIT FOUR The Ordering of Integers

When ordering whole numbers, two different methods are employed which achieve the same results. Procedurally, a number line is drawn, the positions of the natural numbers are mapped in the range in which one wishes to order, and the criteria of the number in the farthest right position as "greater than" the other numbers is used, while the number farthest to the left is considered to be "less than" the others. Zero is the number designating the "least" position of all whole numbers. This ordering is by <u>position</u>, and it is also performed mentally in common situations such as ordering ages (older than, younger than), distances (closer than, farther than), etc. However, in other situations that model real life (prices, lengths of time, weights, etc.), one orders by <u>magnitude</u> (i.e. quantity), where a higher number is considered to be "more than" another amount, a lower number "less than" another (e.g. Sandra has more blocks than Joe has. A box weighing 20 kg is "heavier than" one weighing 4 kg). Zero is the number designating the least magnitude. There is no conflict in W between these two notions (and procedures) for ordering, for by both methods one would conclude that 5 > 3 (5 is "greater than" 3, and 5 is "more than" 3), and 10 < 12.

In Z, however, only one type of ordering exists - that of ordering by position, which procedurally is the same as for W, on a number line which includes positions for the positive and negative numbers and zero. Zero is now greater than all of the negative integers, and less than the positive integers. Ordering may be interpreted as "more positive" and "more negative", or in real-life situations which include negative scales by such terms as "colder than" and "warmer than".

The neutralization model does not order integers in any way, although it does use the quantity ordering which exists for whole numbers since integers represent quantities of chips. A distinction between $^{-}6$ and $^{+}2$ is based on the ordering of 6 and 2 since these numbers designate the quantity of chips in each group regardless of sign. The neutralization of $^{+}18$ and $^{-}18$ is based on the equal amounts of positives and negatives (18 = 18). When adding integers of different signs, for example, $^{+}8$ and $^{-}20$, one decomposes the "larger" one - the designated by the most chips (20 > 8). When one is faced with a subtraction problem such as $^{-}6$ - $^{-}11$, one must notice that 11 > 6, so there are not enough to take away, and this task needs the addition of "zeros". Note that this whole number ordering within an integer environment is in fact ordering by absolute value.

(37) GROUPS OF CHIPS CAN BE ORDERED QUANTITATIVELY

Two (or more) groups of chips (where each group contains chips of one kind only) may be compared to determine which group is larger in quantity (or smaller in quantity). Either counting each group (numerical procedure) or a 1-1 pairing of elements taken from each group where the group with chips left over after the pairing is the "larger" group (non-numerical procedure) may be performed.

Ordering of groups of chips where at least one group contains chips of both signs may introduce an obstacle to the notion of ordering by quantity, since for example the resultant value of 2 negatives may be represented with 2 chips or with 20 chips or 200 chips, etc., so this ordering (by ignoring the neutral elements and comparing canonical values) will not be considered.

(38) INTEGERS MAY BE ORDERED QUANTITATIVELY

One can ignore the sign in front of the integer and focus on the quantity represented by the numeral, in order to determine the "larger" or "smaller" integer. We will call this ordering by <u>absolute value</u>.

(39) ZERO IS AN ARBITRARY ORIGIN, WITH NUMBERS SYMMETRICAL ABOVE AND BELOW THAT POSITION.

Thus far, within the neutralization model, zero has been identified concretely as the neutral element (representational) and as nothing (quantitative). As has been mentioned, one must go outside of this neutralization model to that of the number line in order to develop the notions of ordering within the set of integers. Zero may now be identified as a position on the number line which is a kind of boundary (which can be crossed) separating the negative and positive regions, rather than the "absolute" zero on the whole number line.

(40) POSITIONS ON AN EXTENDED (HORIZONTAL) NUMBER LINE MAY BE REPRESENTED BY INTEGERS

We may label the extended number line with integers, where each integer represents a position relative to zero, and a distance from zero. Opposites are the same distance from zero, just as "back 5" and "forward 5" are equal distances from any starting position.

(41) INTEGERS CAN BE ORDERED BY POSITION ON THE NUMBER LINE, FROM LEAST TO GREATEST

Until this point, there has been no distinction between the comparative size of 3 positive and 3 negative chips, nor of $^{-}10$ and

$^{+}10$. The procedure of ordering in N by designating the number farther right as greater than the number to the left of the first is appropriated for the integer ordering of "greater than" and "less than". This is not a trivial concept when one is to compare $^{-}17$ with $^{-}5$, or zero with $^{-}9$, since one may focus on quantity rather than on position.

(42) DISTINCTION BETWEEN TWO KINDS OF ORDERING OF INTEGERS

There is a difference between choosing a "larger" integer to decompose for additions, and in determining which integer is greater than or more "positive" than another integer.

5.2.5 UNIT FIVE: Abstractions About Integer Operations

(43) REVERSIBILITY OF ADDITION AND SUBTRACTION OF CHIPS

When working with whole numbers, addition combines two groups, and this may be reversed by removal (subtraction) of one of the original groups from the result of addition. Although the procedures for addition and subtraction differ in manipulation, but not in action (add = combine, subtract = remove), these two operations are reversible. Thus with the integer chips, one has

$+ + +$ plus $- - = +$ and $+$ minus $- - = + + +$.

(44) REVERSIBILITY OF THE OPERATIONS OF ADDITION AND SUBTRACTION OF INTEGERS

As with the natural numbers, addition and subtraction of integers as previously defined procedurally, are inverse operations, in that the result of addition may be transformed by subtraction into the initial state. For N, this is illustrated by the two statements 10 + 3 = 13 and 13 - 3 = 10. This reversibility is used in solving equations such as $\square + 9 = 20$ or $8 - \square = 2$, where the equation may be written with the opposite operation (i.e. $20 - 9 = \square$, and $2 + \square = 8$, which then transforms into $8 - 2 = \square$), in order to make the solving process an easier one, by changing a guess and check situation to an operational one. In Z, an example of reversibility is $^{-}10 + {}^{+}3 = {}^{-}7$ and $^{-}7 - {}^{+}3 = {}^{-}10$. Thus one can see that even though these two operations are different procedurally from those in N, the reversibility still holds.

(45) EQUIVALENT TRANSFORMATIONS ON CHIPS

When one has an initial state, I, and desires to reach a final state F, one must perform a transformation on I which will produce F. In N, when restricted to the operations of addition and subtraction, there is only one possible transformation which will suffice. For example:

Initial State	Final State	Transformation	Outcome
7	3	subtract 4	7 - 4 = 3
2	11	add 9	2 + 9 = 11

However, with positive and negative chips, there are always two ways to reach the final state, as illustrated below.

Initial State	Final State	Transformation	Outcome
+ +	+ + + + +	add + + +	+ + + + +
		subtract - - -	+ + + + + ———
- - - -	-	subtract - - -	-
		add + + +	——— - ~~+ + +~~
+ + +	-	add - - - -	~~+ + +~~ ——— -
		subtract + + + +	~~+ + + +~~ -
- - -	NE	subtract - - -	
		add + + +	——— ~~+ + +~~

Note that in the last case, where the neutral element is desired, subtraction leaves "nothing", whereas addition leaves the neutral element.

(46) AWARENESS OF THE EQUIVALENCE OF OPERATIONS ON INTEGERS

With integers, there is no longer a restriction on using a particular operation to achieve a desired outcome. Both addition and subtraction will lead to the same result. It may be noted here that the opposite operation is performed on the opposite integer, as illustrated below.

Initial State	Final State	Operation	Outcome
$^{+}6$	$^{+}1$	add $^{-}5$	$^{+}6 + {}^{-}5 = {}^{+}1$
		subtract $^{+}5$	$^{+}6 - {}^{+}5 = {}^{+}1$
$^{-}2$	$^{+}5$	add $^{+}7$	$^{-}2 + {}^{+}7 = {}^{+}5$
		subtract $^{-}7$	$^{-}2 - {}^{-}7 = {}^{+}5$

This knowledge now gives the student power to make choices in order to simplify certain operations. It may be seen that it is easier procedurally to subtract $^{-}6 - {}^{-}2$ than to add $^{-}6 + {}^{+}2$, and it is also easier procedurally to add $^{-}5 + {}^{-}4$ than it is to subtract $^{-}5 - {}^{+}4$. The traditional subtraction rule of "add the opposite" now may be expanded to include both kinds of transformations of the operation - if it makes the solution process easier, change the operation, and operate with the opposite integer.

The special case(s) of zero as a final state are of note, as these are used in "making zeros" for the addition of opposites, and eliminating equal quantities for the subtraction of integers.

Initial State	Final State	Operation	Outcome
$^{+}6$	0	add $^{-}6$	$^{+}6 + {}^{-}6 = 0$
		subtract $^{+}6$	$^{+}6 - {}^{+}6 = 0$
$^{-}2$	0	add $^{+}2$	$^{-}2 + {}^{+}2 = 0$
		subtract $^{-}2$	$^{-}2 - {}^{-}2 = 0$

(47) AWARENESS OF RESULTS OF OPERATIONS

There must be an awareness that addition does not always "make bigger", and that subtraction does not always "make smaller", and an ability to make sense of a statement like $^{+}8 + ^{-}10 = ^{-}2$, as well as the more "senseless" $^{+}9 - ^{-}5 = ^{+}14$. The criteria for ordering here is that of the traditional number line positions. It is important to not just operate on integers, but to make sense of the results. Since most of the child's previous operations have been within N, there is a strong perspective that the least effect of addition is to leave the number unchanged (5 + 0 = 5), but more usually, it will augment the number (5 + 7 = 12). Now this operation is totally opened up in Z, for addition may result in a) an increase in value b) no change in value c) a decrease in value. The same follows for subtraction. This leads to an awareness that one can decrease or increase an amount by either operation.

5.3 LEVELS OF UNDERSTANDING FOR INTEGERS

The above items in the analysis of integer understanding are listed in order of presentation (i.e. for any notion, the previous notions are considered to be necessary for the understanding of that notion). However, this listing does not examine the levels of understanding which are happening for the student during his learning of integers. Herscovics & Bergeron (1988) have developed a two-tiered (physical and mathematical) model of understanding which is intended to help describe what it means to understand a particular concept in school mathematics. The bare skeleton of this model appears below (Nantais & Herscovics, 1989).

UNDERSTANDING OF PRELIMINARY PHYSICAL CONCEPTS

INTUITIVE UNDERSTANDING	PROCEDURAL UNDERSTANDING	LOGICO-PHYSICAL ABSTRACTION

UNDERSTANDING OF EMERGING MATHEMATICAL CONCEPTS

PROCEDURAL UNDERSTANDING	LOGICO-MATHEMATICAL ABSTRACTION	FORMALIZATION

Understanding of any concept does not proceed linearly within this model, as the opening up of one particular notion may pass back and forth through both physical and mathematical levels, and back and forth within each of those levels. This will be further illustrated later when the integer notions are placed within this model.

The physical level deals with the understanding of non-numerical notions, usually those of an informal nature, or of a particular model (which could be a numerical one) which is used to lead the student into the mathematical notions. In the above integer analysis, this level deals with understanding of the general notions of opposites, as well as the notions about the concrete integer chips. The mathematical level deals with the numerical abstractions of the concept, in our case, the integers themselves.

The following is a brief description of each of the components of this model of understanding:

UNDERSTANDING OF PRELIMINARY PHYSICAL CONCEPTS

Intuitive Understanding: requires a global perception of pre-concepts of the concept to be understood. (e.g. in order to understand that integers are opposites, one must first have a concept of what opposites are in non-numerical situations.

Procedural Understanding: involves non-numerical procedures on physical objects (e.g. a one-to-one matching of chips of opposite sign to form neutral elements).

Logico-Physical Abstraction: is manifest in the construction of logico-physical invariants and generalizations about them (e.g. any resultant value for a group of chips may be expressed in an infinite number of ways).

UNDERSTANDING OF EMERGING MATHEMATICAL CONCEPTS

Procedural Understanding: involves procedures on mathematical objects (in this case, addition of integers for example).

Logico-Mathematical Abstraction: is seen in the construction of logico-mathematical invariants and generalizations about them (e.g. that addition of any pair of opposite integers will result in zero).

Formalization: includes such things as appropriate symbolization of a mathematical idea (e.g. the use of the notation $^{+}3$ and $^{-}6$), and formal definitions or statements or rules governing the mathematical objects (e.g. making an addition rule, declaring zero as the neutral element).

The following chart shows how the integer notions discussed above are placed into this model of understanding.

PHYSICAL UNDERSTANDING: INTEGERS AS OPPOSITES		
INTUITIVE UNDERSTANDING	PROCEDURAL UNDERSTANDING	LOGICO-PHYSICAL ABSTRACTION
(1) oppositeness (5) neutralization of opposites (6a) neutral element	(6b) formation of neutral element (3) resultant value of group of same kind of chips (11) resultant value of group of chips of both kinds (14) addition of chips of like signs (18) addition of equal amounts of chips of opposite signs (21) addition of chips of unlike signs (25) subtraction of chips of like signs when enough (29) subtraction of chips of equal resultant value (33) subtraction of chips when not enough (37) quantitative comparison of chips	(8,9) equivalence classes of neutralization and neutral element (13) equivalence class of resultant values (43) reversibility of addition and subtraction (45) equivalent transformations on chips

EMERGING MATHEMATICAL UNDERSTANDING: INTEGERS AS NUMBERS		
PROCEDURAL UNDERSTANDING	LOGICO-MATHEMATICAL ABSTRACTION	FORMALIZATION
(16) addition of integers with like sign (23) addition of integers with opposite signs (decomposition) (27) subtraction of integers when enough (35) subtraction of integers when not enough to take away (38) quantitative comparison of integers (41) positional ordering of integers	(19) addition of opposites neutralize and result in zero (32) subtraction of equal integers results in zero (42) distinction between "larger" integer and "greater than" ordering (44) reversibility of operations on integers (46) equivalence of operations on integers (47) awareness of results of operations on integers	(2,4,12) integer notation of raised sign with numeral (7) notation of NE as the neutral element (10) +n and -n are opposite integers, where n is a whole number (15,22) integer notation for additions (17) integer addition rule for like signs (20) zero as the neutral element (24) rule for integer addition of unlike signs (26,30,34) integer notation for subtractions (28) subtraction rule for integers when enough to take away (31) identification of zero as "nothing" (36) subtraction rule for integers when not enough to take away (39) zero as origin (40) integers as indicators of position

Thus during the learning of the integer notions, there is continual movement between levels (physical and mathematical) and within levels as well. Not every student will achieve the level of formalization on each topic at the time of initial instruction, but these notions may be formed after much exposure, and consolidated in later grades.

CHAPTER SIX

PRELIMINARY CLASS ASSESSMENT

6.0 SAMPLE PROFILE AND ASSESSMENT PROCEDURE

Through the generosity of two classroom teachers at Pierrefonds Comprehensive High School, and the cooperation of the administration of that school, 53 grade 7 students were made available as possible candidates for the teaching experiment. The number of students in each age group is as follows:

age 11: 2; age 12: 28; age 13: 20; age 14: 2; age 15: 1

In order to assess their previous knowledge and ideas about integers, and their pre-requisite skills both in natural numbers and in notions used in the neutralization model, an initial pretest was designed and then given to the classroom teachers to be administered. This pretest revealed that the students had more knowledge about negative numbers than was anticipated, so a second pretest was designed and administered to probe more deeply and specifically into integer notions and procedures. A third pretest, consisting of one question only, was given to clarify the notion of equivalence classes. Copies of these three pretests, annotated with the rationale for each item, are found in Appendix C. From these results, six 12 year old students were selected to be interviewed, and 4 of these were later chosen for the teaching experiment.

In order to ascertain how these students compared with other students in their age group, 23 grade 7 and 27 grade 6 students at The Study (all female), who were not available as prospects for the teaching experiment, were given the first two pre-tests so that results could be compared. In the grade 7 class, 13 students were 12 years old, and 10 were 13 years old. In the grade 6 class, 22 were 11 years old, 4 were 12 years old, and one was 13.

6.1 CRITERIA USED IN ASSESSMENT

Using the pretests and interviews, the following ideas and skills were assessed.

(a) Children's Ideas about Integers

(1) existence of negative numbers as a result of arithmetic subtraction

(2) existence of negative numbers as a result of subtraction in a real-life situation

(3) existence of negative numbers as positions on a number line, symmetrical about zero to the whole numbers

(4) meaning of integer notation for a negative number

(5) ability to order integers

(6) ability to perform operations of addition and subtraction on a whole number and a negative number

(b) <u>Prerequisite Knowledge and Skills for Neutralization Model</u>

(7) notion of opposites, necessary for identification of integers of equal magnitude but opposite sign

(8) notion of neutrality and neutralization, necessary for the notion of neutralization of opposites, creating a neutral element which has no effect on value, and which is neither positive nor negative

(9) notion of equivalence classes, necessary for notion of equivalence classes of the neutral element and resultant value, and for the procedures which are dependent upon this notion, i.e. removal of equal amounts of opposites to determine resultant value, and addition of neutral elements to allow some subtractions

(10) procedure of decomposition, needed for additions of opposite signs ($^{+}15 + {}^{-}13 = {}^{+}13 + {}^{+}2 + {}^{-}13$) and subtractions of the same sign when there aren't enough to take away ($^{-}23 - {}^{-}30 = {}^{-}23 + {}^{-}7 + {}^{+}7 - {}^{-}30$)

(11) procedure of cancellation (needed for solution of the previous two examples ($^{+}\cancel{13} + {}^{+}2 + {}^{-}\cancel{13} = {}^{+}2$ and ${}^{-}\cancel{23} + {}^{-}\cancel{7} + {}^{+}7 - {}^{-}\cancel{30} = {}^{+}7$)

(12) regrouping by commutativity, to be able to combine integers that may be cancelled due to neutralization.

6.2 RESULTS OF CLASS ASSESSMENTS:

In order to simplify the reporting of results from the three groups of students, the Pierrefonds grade 7 group will be reported as

PF, the grade 7 class from The Study as S7, and the grade 6 group from The Study as S6.

6.2.1 Results of Individual Pretest Items

(1) EXISTENCE OF NEGATIVE NUMBERS AS A RESULT OF SUBTRACTION

Question 1 (f) gave the task $2 - 8 = \square$

Results: The two most common answers were ⁻6 and 6, with the breakdown as follows:

solutions	PF	S7	S6
- 6	51%	24%	31%
6	31%	62%	58%

It must be said that for those students who were able to give -6 as the result of this subtraction, there is not enough evidence from this type of task to reveal whether the -6 exists for them as a negative number, or if it is just a notation for the result of the subtraction of a larger number from a smaller one.

(2) EXISTENCE OF NEGATIVE NUMBERS AS A RESULT OF SUBTRACTION IN A REAL LIFE SITUATION

Question 8 put the subtraction of whole numbers resulting in a negative number into a temperature situation, and reads as follows:

> The thermometer reading on Dec. 1 was 5 degrees. During the night, the temperature fell by 8 degrees. What was the reading on the thermometer at that time?

Results: The answer -3 was obtained by an overwhelming majority of the grade 7 students (PF: 82%, S7: 86% , and S6: 62%), but there is no evidence of whether this notation simply means "3 below zero", a statement using a whole number, or whether the temperature "minus 3" is a negative number in their understanding.

(3) EXISTENCE OF NEGATIVE NUMBERS ON A NUMBER LINE

Question 16 asked the students to put the numbers $^{-}1$, 7, 0, 5, $^{-}7$ on a number line.

Results: A number of students at both schools simply re-wrote the numbers in ascending order rather than on a number line, and if they were in the correct order, they were counted as a correct answer. Again responses were good: 78% at PF, only one erroneous answer for S7, and 74% of S6. From these one hundred students, only 12 answered incorrectly, with a variety of solutions. Although their classroom teachers had stated that there had been no previous integer exposure, these results seem to indicate that most children were familiar with the positions on the integer number line.

(4) MEANING OF INTEGER NOTATION FOR A NEGATIVE NUMBER

Question 6 tested the student's familiarity with integer notation, in reference to a specific integer, worded as follows:

> Have you ever seen a number like -5 before? Sometimes it is written as (-5) or as -5.
> (Put an x): Yes ☐ No ☐
> If you have, what do you think it means?

Results: The following responses were given (note that some students wrote more than one meaning, and that numbers given represent number of students):

DESCRIPTION	PF	S7	S6
negative 5	14/49	6/23	3/27
minus 5	1	3	
5 below zero*	8	2	8
"negative"	1	2	3
below zero**	11	5	2
a temperature	1	4	5
a score if you're bad			1
you have counted back from zero		1	1
subtraction result	5	1	1

to subtract 5	1	1	
5 less items	1		
something missing	1		
(no meaning)	7	1	6

* also includes 5 under zero, 5 lower than zero, 5 less than zero

** also includes under zero, lower than zero, less than zero

Note that only 14% of the students had not yet constructed any meaning for an isolated negative number, and that all meanings given were correct (yet some incomplete).

(5) ABILITY TO ORDER INTEGERS

Question 13 asks the student to order (a) two negative numbers, (b) a negative number and its opposite positive number and (c) a negative number and zero. (Proper use of the > and < signs was tested in question 2, where only 4 students gave incorrect answers.). The question reads:

Use the symbol > or < for these comparisons:

(a) -5 ___ -1 (b) 6 ___ -6 (c) -2 ___ 0

Results: There were very few students, even in grade 6, who had difficulty with this question, with task (a) having the poorest results, with success rates of 82% for PF, 87% for S7, and 81% for S6.

(6) ABILITY TO PERFORM THE OPERATIONS OF ADDITION AND SUBTRACTION ON A WHOLE NUMBER AND A NEGATIVE NUMBER

Question 15 gave a series of five tasks involving the operations of addition and subtraction of a negative number and a whole number. The following shows the results for these tasks. Note that the first answer given for each question is the correct one.

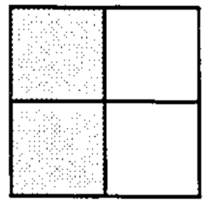
QUESTION	ANSWER	PF	S7	S6
(a) 2 + -7 =	- 5	50%	52%	59%
whole plus neg.	- 9	32%	30%	19%
(b) -6 + 1 =	- 5	56%	52%	67%
neg. plus whole	- 7	32%	43%	19%
(c) -6 - 4 =	-1 0	44%	35%	52%
neg. minus whole	- 2	48%	48%	26%
(d) 10 + -2 =	8	48%	39%	63%
large whole plus	-1 2	22%	26%	11%
negative	-8	16%	1 student	1 student
(e) 4 - -1 =	5	1 student	1 student	1 student
whole minus neg.	3	36%	30%	37%
	$^-3$	26%	30%	15%
	- 5	22%	30%	19%

It must be noted that approximately half of the students could successfully perform additions with apparently no formal instruction. It appears that the most common errors were made by performing the indicated operation, and then putting a negative sign on the answer (for example, for 2 + -7, one would add 2 plus 7 to get 9, then put a negative sign in front of the 9). Note here that it was an oversight on the author's part to not superscript the negative signs particularly on these operational tasks, as one cannot evaluate the interpretation that the students may have placed on for example 2 + -7 when compared to 2 + $^-7$. There is the possibility that this could be interpreted as a confusion of two juxtaposed operations, rather than an operation with a negative number.

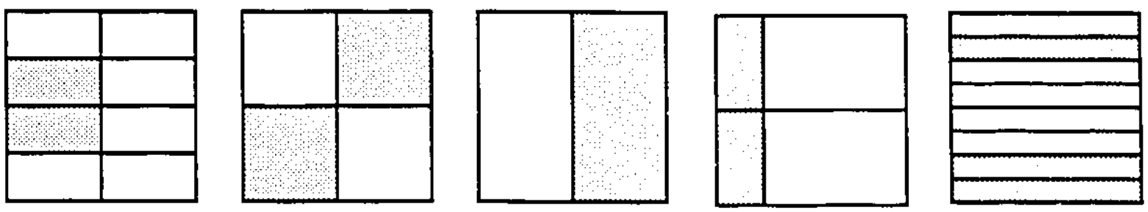
(7) NOTION OF EQUIVALENCE CLASSES

The following task (question 5) was designed to test the notion of equivalence in a non-numerical way, but poor results (PF: 57%, S7: 66% and S6: 65%) seemed to indicate problems with the interpretation of the equivalence of the diagrams (for example, the fourth diagram was chosen as equivalent to the original one by 27 students, as many students wrote 2/4 beside it).

<u>Some</u> of the following diagrams represent the same fraction as



Write YES below the diagrams that show the same fraction.



Thus the following (question 18) was given to the students at Pierrefonds to verify that the concept of equivalent value was understood. The idea for this question is from Artzt and Newman (1991).

In front of you, there is a large container filled with pennies, nickels, dimes and quarters. You must give your teacher 20 cents. What coins will you give him? ______________. How many other ways can you find to make 20 cents? List them.

Results: Students gave either 8 or 9 different ways to make 20 cents, except for the weaker students, who saw 4 or 5, but all students without exception gave evidence of the notion of equivalence within this particular context of equivalent value.

(8) NOTION OF NEUTRALITY AND NEUTRALIZATION

The neutralization model depends entirely upon the notion of a neutral state or neutral element having no value and no effect, being neither positive nor negative. Therefore questions 9 and 10 were given in order to discover the child's intuitive sense of neutralization. It must be said that a question more appropriate to the neutralization model within a real life (science-based) context could be based on the neutralization of acids and bases, or the idea of positive and negative charges, but the grade 7 student has not yet encountered these ideas in the curriculum. It was difficult to design tasks that carried the desired notions of neutral and neutralize, especially in a written rather than interview environment, and in fact, the following questions which were given are semantically inappropriate.

(9) Joy had a sore throat, so her mother prepared a hot drink of water with lemon for her. Joy tried it and found that it tasted very sour. Her mother added some honey to neutralize the taste.
What does the underlined word mean?

(10). Premier Bourassa wants to hold a referendum (a vote where everyone chooses either yes or no) to see if the people who live in Quebec want to stay in Canada, or if they want Quebec to be a separate country. Everyone will get a chance to vote yes or no for separation. The grade 11 classes discussed the issue, since some of them are old enough to vote. Sandra said that she would vote no - she wants Quebec to stay in Canada. Evelyn would vote yes - she thinks Quebec should be separate. Jerry is neutral about the issue.
What does the underlined sentence mean?

Results: For question 9, the child's meaning of the action "to neutralize" was sought. Some of the answers were difficult to classify, as "to sweeten" could mean "to make sweet" or "to sweeten the sour to make it not sour", while others such as "to make the drink taste better" give no indication of whether or not they understood neutralize, because it can be seen as a <u>result</u> of the neutralization. Examples of clearly acceptable answers are "kill the taste of the lemon", "to break down the sour taste", "to even out, or balance, the taste", "to make it not too sweet, not too sour", etc. Answers of this type were given by 38% of the PF students, 43% of the S7 students, and 42% of the S6 students.

For question 10, the concept of being neutral was investigated. Acceptable answers were of the type "is not taking any sides", "has no opinion", "not sure", "doesn't mind", etc. For the PF group, 67% had this notion, while at The Study, 71% of the grade 7's and 62% of the grade 6's gave acceptable answers.

(9) NOTION OF OPPOSITES

The following task was designed to verify that opposites were seen as pairs, opposite in nature, and of equal magnitude:

Write the opposite of the following:

tall ____________ 5 steps up ____________

10 degrees below zero _________ succeed _________

Results: There were no difficulties, except that for "5 steps up" and "10 degrees below zero", between 30% and 40% of students did not write a numeral with their answer, instead just writing "steps down" and "degrees below zero". In order to ascertain whether this was a problem with the nature of opposites when quantity is involved, or whether combining a numeral with words was at issue, the opposite of "seven kilometers east" was asked for (question 14a) in the second pretest. This particular problem did not emerge in this second try, and 82% of all students were successful at the task.

In order to discern if the students had the notion of negative integers as being the opposite of positive integers, question 14(b) asked for the opposite of 5. Sixty-six percent of PF students, 65% of S7 and 59% of S6 gave -5 as the solution. In fact, since the number given to the child is a whole number (5) as opposed to a positive integer (-5), it is difficult from a paper and pencil task to assess what "opposite" meant to them in this context, although a conjecture is that it may have been thought of as a position on a number line opposite to 5.

(10) PROCEDURE OF DECOMPOSITION

Questions 1 (d) and (e) tested to see if given a sum and one addend, the student could find the other addend either by subtraction or by adding on from the given addend.

(d) $43 = 13 + \square$ (e) $62 = \square + 19$

Results: Part (d) was answered correctly by 94% and 92% of the grade 7's, while the grade 6 class were 100% successful. Part (e) had a slightly lower success rate, of 86% and 85% for the grade 7's and 95% for the grade 6 students. Eight percent of the answers had errors that were arithmetic in nature.

(11) PROCEDURE OF CANCELLATION

Question 17(a) (Put the missing number in the box: $485 - 376 + 57 + 376 = \square$) was designed to ascertain whether a student would perceive an easier way to obtain the answer to a string of operations, and employ it. This would be necessary for future integer cancellation tasks. There was no use of the word "cancellation" here, but rather this was a task to assess the student's inclination to cancel.

Results: From the 97 students given this question, only 3 (all from PF) gave evidence of cancellation (one drew lines through both 376's and all three showed the addition of only 485 + 57). A total of 39 students showed no work at all or had erased their work. Most of the recorded errors were arithmetic in nature, but 4 students showed difficulties with the minus sign, as follows: (a) 2 students added the last three numbers and subtracted the leading number from the sum, demonstrating what Herscovics & Linchevski (1991-a) have termed "detachment of the minus sign" from the 376, as well as a mental bracketing of the last three terms, which yielded a situation of having to subtract a larger number from a smaller one (b) one student added 376 + 57, then subtracted this sum from 485, then added the last 376 (detachment of the minus sign) (c) one student added 485 + 376, then subtracted 57, then added 376 (mixing of operations).

Since many children did not show work, there is not enough evidence to show how frequently cancellation is used when possible, nor does this task uncover student's beliefs about the equivalence of results when cancelling is performed, nor about the ability to cancel when a subtraction appears before an addition of the same amount.

(12) REGROUPING BY COMMUTATIVITY

In order to notice and perform the above cancellation, the student must be able to look globally at the sentence, so solutions to the above problem were analyzed for the prerequisite of making rearrangements in the order of performing the operations (even if cancellation was not performed), as opposed to solving from left to right. Only 12 of the 58 students who showed their work and did not cancel showed rearrangement of the numbers, half of them pairing the first two terms, the second two terms, and adding both results, and 5 of them grouping the additions, either before or after subtraction. Forty-three students (74% of those who showed work) showed the performance of operations from left to right, while 26% (12 + 3) rearranged or cancelled. Since integer cancellation must be noticed in a string, these results indicate that intervention must be given initially in the context of whole number strings.

6.2.2 Discussion of Results of Pretests

Although it was originally felt that the Pierrefonds students had too much information already about negative numbers and integers, the students at The Study, even in grade 6, also seemed to have developed some strengths, although sometimes in different areas. The grade 6 students often surpassed those in grade 7 at their school, perhaps being a stronger class, or perhaps having better intuition, and being less burdened by formal knowledge.

Certainly integer notation for negative numbers was recognized and used by a majority of the students in all grades, particularly in a real-life temperature context. Documents given to the students did not have a raised negative sign, and none of the students used one on their own. Half of the PF students responded with a negative number to the subtraction of 2 - 8, and most students could draw the extended number line or order a group of positive and negative numbers. About 60% of the students gave -5 as the opposite of 5, and a variety of meanings were suggested for a particular negative number.

Since the planned mathematization of integer operations involves cancellations within an expression, the scanty evidence of this procedure is a concern.

Of interest was the proficiency at operations involving a negative number and a whole number, many tasks being solved by at least 50% of the students, despite having apparently received no formal instruction in the school environment. Even more interesting was that the group who were most successful were the grade 6 students.

This assessment was designed to give some evidence of how much exposure grade 7 students had already had to negative numbers, and what kinds of tasks they appeared to be proficient at, although this in no way can be expected to reveal what kind of understanding these students had constructed about integers. Questions evoking a "thinking" response were not included in this paper and pencil environment, as it was felt to be preferable to pursue this nature of questioning in the format of the individual interview, although the disadvantage of this approach is that it limits the number of subjects that could reasonably included in the scope of this thesis.

6.3 INTERVIEW DESIGN

In order to probe deeper into the children's thinking and understanding about integers, a semi-standardized interview was designed, and carried out using 6 of the 12 year old students at Pierrefonds Comprehensive. Tasks involving cancellation were also included, since so few students had given evidence of this technique on the pretest, and since many students had given no indication of their strategy of solution. One of the types of information sought from the interview was to find out to what extent any previous instruction had been encountered, and to what extent the students had built their own mental models of integers (or negative numbers) from other exposure to them or from intuitive reasoning about them.

The following students (all 12 years of age) were chosen for the interviews:

(1) Robert, who appeared to be able to operate with integers, but who seemed to be weaker in integer notions. His teacher had labelled him a top student.

(2) Steven, who appeared to be a strong student in both integer knowledge and operations. His teacher had labelled him average.

(3) Jason, who seemed competent with whole numbers, but could not perform integer operations. He seemed inconsistent with integer notions. His teacher designated him as average.

(4) Richard, who could not correctly draw an integer number line, nor perform integer operations (except for 10 + -2). His teacher designated him as a top student.

(5) Connie, who made numerous errors in the integer tasks, even in ordering and drawing a number line. Her teacher labelled her an average student.

(6) Marilyn, who did not attempt the integer operations, and had no meaning for -5. Although her teacher had designated her as average, he was surprised that she was considered for the interview, stating that she had some difficulties in math.

The design of the interview that was given to these 6 students is found in Appendix D, with the planned dialogue underlined. Each student was given worksheets with the tasks only, and both the interviewer and the observer (Dr. Anna Sierpinska) had the detailed observation sheets.

6.3.1 Results of Individual Interview Items

(1) EXISTENCE OF NEGATIVE NUMBERS AS A RESULT OF SUBTRACTION

On the pretest, all six students had given -6 as the answer to 2 - 8. Item 7 (a), 6 - 15 = $\square$, was given to examine the nature of their reasoning.

Results: Marilyn said she couldn't do it because "6 is smaller than 15", and Jason said the answer is 9, since you can't do the subtraction, you have to "switch it around". He had also "switched around" the subtraction of 322 - 814 which had appeared as a result of an incorrect procedure in a previous whole number task. The other students gave the correct answer, but neither Steven nor Richard could give a reason for their answer, although Steven had been confident that the answer was "obvious", then became unsure when asked for the reason. Connie said it would be a negative, because you can't subtract, and Robert's reasoning was that "it's the opposite" of regular subtraction.

Robert and Connie showed some form of procedural understanding, and Steven had perhaps initially had an intuitive approach that he could provide no reason for. Marilyn showed that she has not moved beyond the environment of the natural numbers, and Jason had not moved into the negatives for this task, although as will be shown later, he easily uses the integer number line.

(2) NEGATIVE NUMBERS AS A RESULT OF REAL-LIFE SUBTRACTION

This was not tested in the interview, and Connie was the only student who did not get the correct answer in the pretest (writing 3 instead of -3).

(3) NEGATIVE NUMBERS AS POSITIONS ON A NUMBER LINE

The students were told to guide the interviewer in the making of a number line and labelling of the numbers $^{-}6$, 9, 2, and $^{-}4$.

Results: Jason, Robert and Steven had no difficulties with the task of drawing a number line both on the pretest and in the interview. The other three students had drawn an incorrect number line on the pretest, but in the interview, their number lines were correct. Connie and Jason allowed irregular spacing of the numbers.
Four students spoke of zero as the center of the line, and of the line being divided in half by this position. Most spoke of negative and positive sides.

(4) MEANING OF INTEGER NOTATION FOR A NEGATIVE NUMBER

In this interview, the students were asked for the meaning of the written symbol -32 (in the pretest, -5 had been used).

Results: For Richard, the negative sign meant it's under zero. Marilyn had no meaning for it, and Robert saw it as representing something that's missing. Connie knew that some subtractions yielded a negative number as an answer, "even if you do it by a calculator". Jason stated that it's below zero, and that it represents a missing quantity. Steven had the most meanings, giving it the name "negative 5", showing it as a result of subtraction, and stating that it's under zero.

All of these responses reveal an understanding of a negative number as a position on a number line, or as the result of an arithmetic operation on whole numbers.

(5) ABILITY TO ORDER INTEGERS AND MEANING OF THE ORDERING

The students were asked the following in order to evaluate their strategy for ordering negative numbers:

(a) One of your classmates asks you: "Which of these two numbers is larger?" (-10, -8) What would you tell him? How would you convince him if he didn't believe your answer?

(b) Use one of these signs (> or <) in the blank:
-45 ____ -920
How did you decide which sign to use?

Results: All but Connie were able to order integers, using the idea that the negative number "closer to zero" is larger. (Unfortunately, they were not asked what "larger" meant to them in this context, nor why this criteria sufficed.) Connie ordered by absolute value, removing the signs and looking only at the number, in a sense, ignoring the existence of the negatives. Steven felt that there are two ways to order, stating, "it's smaller in a sense, but it's greater in a sense. It's greater below zero, but it's less above.". For all but Connie, zero was the reference point for ordering both positives and negatives separately in a number line setting, Steven and Robert explaining that you use opposite rules for opposite kinds of numbers. Unfortunately, only the comparison of two negative numbers was tested.

(6) OPERATIONS OF ADDITION AND SUBTRACTION ON A WHOLE NUMBER AND A NEGATIVE NUMBER

The second pretest contained the following tasks (question 15):

(a) $2 + -7 = \square$ (b) $-6 + 1 = \square$
(c) $-6 - 4 = \square$ (d) $10 + -2 = \square$
(e) $4 - -1 = \square$

while the interview included the following (question 7) in order to ascertain the child's reasoning process for solution:

(b) $21 + -21 = \square$ (c) $-4 - 16 = \square$
(d) $-5 - -5 = \square$ (e) $13 + -1 = \square$
(f) $-9 + 7 = \square$ (g) $3 + -10 = \square$
(h) $20 - -6 = \square$

Results: The notation here did not include a superscripted sign, and although it was a concern that on the pre-test this may have caused confusion between the two juxtaposed signs, each of the six students read the task in a way that showed identification with the negative sign (for example "twenty-one add negative twenty-one").

Marilyn had not attempted the questions on the pretest, and in the interview clearly stated that she had no idea what to do if the negative sign had to be taken into account, otherwise she would ignore the sign entirely. She did, however, give zero as the answer to (d), stating that the numbers were the same, and <u>both</u> negative, in effect performing whole number subtraction. She had no awareness of negative numbers as <u>numbers</u> which could be operated on or with.

On the pretest, Richard had obtained the common incorrect answers for the first 3, and the correct one for the fourth. In the interview, he frequently changed his answers, and clearly had no strategy for any of them, admitting to guessing for some, except for (b) which he saw as the addition of a positive and a negative of the "same number", and gave zero as the answer (but was not asked why). His answers did not match with those of the pretest for similar questions. He did not attempt the subtraction in (h).

Both Connie and Jason obtained the common incorrect answers on the pretest (appearing to have performed the operation on the two numbers disregarding the sign of the negative integer, and attaching a negative sign to the answer). However, in the interview, neither one used this strategy. Connie began (b) by adding 21 and 21, but did not know what sign to put. Then she decided that -21 "is like zero to me" so the answer had to be 21, since there was nothing added on. She adopted this technique to solve all but (d), rearranging the questions so that the positive number came first, then writing it as the answer, since the negative number did not exist for her as a <u>number</u>, (only as a position on a number line), and could not be added or subtracted. She gave the answer to (d) as -0, because both of the numbers were negative, but when questioned about zero being negative, she said it (zero) didn't really exist. It seems that only numbers that represent a quantity exist for her. Jason also displayed this concept that negative numbers are "sort of nothing", so they can't be added on, but if the question <u>began</u> with a negative, he was able to either "subtract down" (c), or "add up to increase it" (f), using a number line positional procedure. He had no problems with (d).

Steven had correctly answered all but the subtraction question on the pretest. However, in the interview he seemed to be confused, and possibly should not have been given the more difficult ones first. He said that sometimes his parents had given him some like these to solve (although he said he had not been taught), and that he had constructed his own procedural rule, which he knew worked only because his parents marked the answers right. This rule was "I change my additions to subtractions", because "it's the opposite of above zero". Thus Steven never adds, ("you can add, but I don't know how to"), only subtracts. He was able to perform all but one of the addition tasks correctly using his rule, yet on reflection, he changed two of his answers (d and c) when he tried to think logically about the operations. He was confused with subtraction example (h), stating that if it had been -20 minus 6, he knew it would be -26. Thus he has some procedural techniques, but his confusion indicates his lack of understanding of operations on and with integers.

Robert correctly solved all problems both on the pretest and in the interview, but prefaced each answer with "I think" or "it should be". When the negative number was larger than the positive number (g and f) he rearranged so that the negative came first, then added the positive to it, coming "closer to the zero". When the positive number was larger (e) or equal to the negative (b), he saw it as "something missing" from the positive number, so he subtracted. He used a number line movement for (c), and regarded (d) as a regular subtraction. He was able to obtain the correct answer for (h) by a combination of a rule (it's like adding instead of subtracting since negative is the opposite of positive), and justification on the number line of adding the "number of spaces" of 20 above zero to those of 6 below zero. His understanding of a negative number is "something missing", and positional.

In summary, it may be seen that those who could not solve correctly seemed to be fishing for a rule or relationship that would work for most or all examples. Those who were successful, for some or all, used a different strategy according to the nature of the example, some number line based, while others were based on their perceptions about numbers. Human & Murray (1987) report similar "non-concrete intuitions" used by upper primary school students, who had no instruction about integers. They termed this "analogical methods of reasoning", since the students used notions about whole numbers (for example, integers are opposites to positive numbers, so they behave in an opposite way - instead of subtracting, one can add) as a basis for their strategies.

(7) NOTION OF EQUIVALENCE CLASSES

This was not tested in the interview.

(8) NOTIONS OF NEUTRALITY AND NEUTRALIZATION

This was not tested in the interview, as the pre-test revealed that these notions would need to be strengthened, and the concept approached in the context of the neutralization model.

(9) NOTION OF OPPOSITES, AND INTEGERS AS OPPOSITES

The students were asked to give the opposite of (a) rich, (b) negative and (c) 20 steps back. They were asked to explain what made their responses opposite, and for the last task, they were asked if 8 steps forward could be considered to be opposite to 20 steps back, in order to determine the significance of magnitude.

They were also asked if they felt that numbers could have opposites, and what they thought the opposites of (a) 700 and (b) 4 were.

Results Everyone was able to give a good explanation of opposites, except for Marilyn, who could give an opposite, but not explain what made it opposite. Richard felt that the opposite of 20 steps back could be zero steps forward, and there is logic in his answer (i.e. movement, no movement). On the subject of opposites of numbers, one may note that only Steven and Marilyn consistently believed that a negative number is the opposite of a natural number (the question had not been given with positive numbers). Jason stated that numbers do not have opposites, and his attempt to suggest an opposite for 700 was to reverse the digits (007). The other students were inconsistent in their responses, giving a negative number as the opposite of a natural in one case, then giving a different type of response in another. For example, Robert gave -5 as the opposite of 5, -700 as the opposite of 700 (you have 700, and you're missing 700) and 8 as the opposite of 4 (because it's twice the size of 4).

Although Richard had written -5 as the opposite of 5 on the pretest, in the interview he stated that nothing exists that is the opposite of a number. This question may not be of value, since there may be a confusion over natural and positive numbers - for some children they may be the "same" when in an integer environment, but this was a apparent natural number task. There could also be an added confusion over the interpretation of the term "opposite" for those children who responded differently to what the interviewer saw as similarly worded tasks.

(10) PROCEDURE OF DECOMPOSITION

None of these students had difficulty with this task on the pretest, and it was not discussed in the interview.

(11) PROCEDURE OF CANCELLATION

Steven was the only student from this group who had cancelled in the pretest (485 - 376 + 57 + 376 = $\square$). The first task in the interview (1 a,b,c) was designed to determine when a student would cancel, and what would prompt the cancellation. If the first and second were solved without any cancellation, the student was asked to consider if there would be an easier way to solve (b) before they were given task (c).

(a) 392 - 143 + 85 + 143 = $\square$
(b) 25 + 814 + 322 - 814 = $\square$
(c) 647 + 299 - 299 = $\square$

Steven began to solve the first problem by rearranging the numbers to make the additions easier - he stated that he would subtract 143 from 392, then "add the bigger one, then add the smaller one". He performed the subtraction, then noticed that he would be adding on the same number he had just subtracted. At this point he said that to make it easier he could just add 392 plus 57, but first he continued to carry out his original plan. When he actually performed the simplified operation, he subtracted rather than added. In the second example, he immediately spotted the two 814's, but because the order of the operations performed on this number was reversed (add then subtract), he wasn't sure that he could cancel, so he tried to judge the outcomes, then tried it both ways. For (c) he spontaneously cancelled with confidence. His "discovery" of cancellation and his insecurity about cancellation in the second task indicate that his cancellation on the pre-test may not have been a usual strategy.

Robert was the only student who never considered cancellation, nor noticed that it might be an option even in the simplest example, as his strategy was to work sequentially from left to right. Even when asked to consider an easier way to solve the second one, he couldn't see anything. He made an arithmetic error in the last one, and would not have noticed the cancellation until he was asked to check his answer on the calculator. When he saw the leading number appear on the display, he finally noticed the opportunity to cancel, then saw that he could have cancelled in the other two, although he was not sure whether the remaining operation in the first one should be addition or subtraction.

Connie was the only student who, not having spontaneously cancelled on the first or second, saw it in the second example when asked to look for an easier way. However, when asked to consider an easier way to solve the first example, she said she would add both 143's, subtract that from 392, then add 57. She spontaneously cancelled on the third example.

Richard did not see the cancellation when asked to find an easier way for the second example, but spontaneously cancelled in the third. He then said that it was possible in the second example, but not in the first, because the operations were in the reverse order, and you must add 143 and 143 (he had initially solved incorrectly by rearranging, then adding 143 + 143, then adding 57, and this total was subtracted from 397).

Jason did not see the cancellation until he spontaneously cancelled in the third example. When then asked to consider an easier way for the second, he said he would multiply 814 by 2, then add everything, and like Richard, for the first example he would combine the two 143's and the 57, and subtract from 392.

Marilyn saw in the third example that she could first do 299 minus 299, which would give her zero, then she would add zero to

647. She was the only one who seemed to see this "cancellation" as a separate operation to be performed and included in the solution of the problem. She then did the same for the second example, but for the first example, she began to have a problem with the signs, since she wanted to do 143 minus 143, then add it to the others, but she became unsure about the legitimacy of the combination of the 143's since the signs were in the opposite order.

All six students demonstrated a knowledge of the procedure of cancellation (when a number is added, and then subtracted, one may omit the performance of these two operations, since that particular outcome is zero), at least when given the third example. They all showed some degree of confidence that these two numbers may be ignored since they will have no effect on the answer to the problem. However, it seems that there are four difficulties that emerged as these students were faced with situations where cancellation would have made the task easier, yet cancellation was not spontaneously used by anyone other than Steven.

(1) They must notice that there is a possibility of cancellation (i.e. that in a string, two equal numbers preceded by opposite operations may be cancelled and omitted from the string). Robert's step by step procedural approach seems to have obscured the possibility, so what is needed is a global approach to the task, and some of the students showed this by rearranging the numbers to make their solving easier, yet their rearrangements did not enhance cancellation.

(2) They must trust that the outcome of the entire task will be the same when this cancellation is performed. This means that they must have insight into the outcome without having proof of it.

(3) They must know that the inverse situation is also true, that if a number is first subtracted and then added, the effect is the same as adding or subtracting zero (the fact that so few students showed evidence of cancellation on the pretest may be due to this arrangement). This is more difficult in the realm of whole numbers because they must be aware of the number preceding the subtraction, so that they can sense that it is being taken away from something, then added back onto it. The case of addition then

subtraction may be thought about on its own, since one can start with that first number with no regard to what preceded it.

(4) They must have knowledge of what operations are left behind when the cancelling is done. Of those who felt that the numbers in (a) could be cancelled, there was confusion about whether to add or subtract the remaining two numbers. It appears that the student cancels only the number, and not the operation assigned to it. Herscovics & Linchevski (1991-b) also observed this phenomenon of using the sign following the leading number after cancellation, and termed it "jumping off with the posterior position".

(12) REGROUPING BY COMMUTATIVITY

Since rearrangement of the numbers is what led Steven to notice cancellation in the first example, the solving styles used by these students in the cancellation tasks is of interest. In both the pretest and interview, Connie and Robert solved from left to right, although when Connie was asked to look for another way, she was jolted from this approach which revealed the cancellation in one example, but rearrangement of another example did not. On the pretest, Jason first subtracted, then combined the additions, and here he paired from left to right (i.e. combined first 2, last two, then added the results) on the first two, which worked for him in the first example, but not in the second, since it involved performing 322 - 814 which he switched to 814 - 322. Marilyn and Richard rearranged in the first example, Marilyn correctly pairing (as she had on the pretest), and Richard incorrectly combining the additions, then subtracting the total, but they both solved from left to right in the second example. Richard had solved from left to right on the pretest.

Thus Steven, Jason, Richard and Marilyn all spontaneously rearranged to solve at least one task and Connie rearranged one she had previously solved sequentially, but Connie, Jason and Richard each had errors due to their rearrangements. Instructional intervention must take into consideration not only a global view of a string of operations, but also an awareness of justifiable and unjustifiable groupings.

6.3.2 Discussion of Results of Interviews

The following notions, incomplete notions and misconceptions of integers held by these six students were observed. Some of these notions were not solid, and were often observed to be changing or developing during the interview.

(a) The notion of negative numbers in general as being opposite to (and distinct from) positive numbers in general, with the notion that negative numbers need opposite strategies from the positive numbers, evidenced in (i) the use of an "opposite" rule for ordering negatives ("closer to zero" rather than the "farther from zero" used for the positives) and (ii) the justification of Robert and Steven that you can subtract to find the result of addition since subtraction and addition are opposite operations, and "it's the opposite with negatives" (Robert also used this to add to find the result of 20 - $^{-}6$). This notion could be an obstacle to the acceptance of integers as a set of numbers containing both positive and negative elements.

This notion of the oppositeness of negatives and positives does not appear to include the notion of a particular negative number being the opposite of a particular positive number except for being positioned "above" and "below" zero.

(b) The notion of negatives as positions on a number line was not only seen in the correct drawings of integer number lines, but also in the strategies of Robert and Jason who could start at a negative number and add up or subtract down from a negative number. A couple of the students mentioned having seen the number line before in school. This facility with the number line allows for procedural facility, but does not give any meaning to the number.

(c) The notion of negatives as nothing, nonexistent, (explicitly stated by both Jason and Connie), and therefore unable to be added to or subtracted from a natural number. Connie and Marilyn in particular denied their existence both in their approach to operational tasks and in Connie's strategy for ordering integers.

(d) The abstract nature of the negative numbers, since they did not represent a quantity, but position only.

The interviews shed some light on the level of difficulty of addition and subtraction tasks. We had seen from the pretests that tasks which began with a negative number followed by an operation with a positive number were just as well answered as tasks which began with a positive number followed by an operation with a negative number, and from the interviews one can see that in general, the students who solved the second type simply rearranged them into the first type, while those who could not solve them could not operate <u>with</u> a negative number (not yet having a quantity representation of it, even though positional status had been granted to the negative numbers). Thus, the observations made from pretest results and from the above suggest that students do bring number line notions to integer tasks, but only notions that are in agreement with the natural numbers, and do not conflict with them. This may or may not prove to be an obstacle to the acceptance of integers as a new set of numbers if the number line is used as the integer model.

6.3.3 <u>Individual Student Assessment</u>

Steven had a formal, rule-based approach for all operations presented to him, but his confusion when he began to reason about the operations and their results show that he lacked understanding of what it means to operate with and on negative numbers. He had no concrete model to back up his "rules", and could not justify his answers. His notion of the oppositeness of integers was an "above" and "below" zero frame of reference. He was the most "dynamic" student, involved in each task, trying different approaches, testing his ideas, and aware of relationships. He was articulate about every step of his thinking.

Robert did many of the calculations on large numbers in his head, was aware of his mental processes, yet was not innovative, always using the same procedure. He was able to make sense of the negative number tasks by number line reasoning, and by the notion that negative numbers are the "exact" opposites of positive numbers. His notion of oppositeness was a "have" / "missing" concept.

Jason discriminated between different types of negative number tasks based on whether the task began with a negative number (position), or whether a negative number was to be added to or subtracted from a whole number. He had no meaning for the latter, since "this is below zero, so you're adding sort of nothing" (quantity). His notion of opposites was "above" and "below" zero, and he called these "regular" and "negative" numbers, implying that he used an extended number line rather than an integer line. For an exclusively whole number task (6 - 15 and 322 - 814) he reversed the numbers and subtracted rather than descending into the negative range of the number line. He was willing to extend his thinking when he did not know an answer (reversing the digits of 700 to come up with an opposite), yet accepted his limitations in the integer tasks.

Connie initially approached the tasks with operations on negative number by performing the designated operation on whole numbers, then deciding which sign to put, but then she began to reason about operating with negatives, and stated that the operations did not make sense, since "that is really zero to me". She also removed the negative signs for ordering tasks, implying that negative numbers do not exist for her in a quantity context. Negative numbers existed only as the result of some subtractions, and her description of the oppositeness of negative and positive numbers was that they were the results of subtractions or additions respectively. Connie did not notice things until her attention was specifically directed to them, seeming to focus on one thing at a time, like Robert, but not always focusing on the most appropriate thing. She was not always aware of her thinking processes, but would just give an answer that she could not explain. She was very inconsistent, sometimes showing good understanding, and other times having no rationale for her statements.

Marilyn showed the least knowledge of negative numbers, only that they were below zero (position), but her reasoning was mature for the tasks that she could perform. She did not attempt any question for which she had not yet formed an idea or procedure. She could do none of the integer operational tasks, because she did not know if "this little sign" meant anything special in the context of operations. She was not chosen for the teaching experiment only because her teacher had conveyed that she was a very weak student who had difficulties in math.

Richard seemed very direct, giving definite answers when he knew them, but quickly guessing when he did not know. He was unable to offer any explanation of his thinking. He did not seem to be very "interesting" as a candidate for the teaching experiment, since he did not seem to spend much time thinking about the tasks presented.

Based on the interviews and the pretest, four students were chosen to be used for the teaching experiment: Steven, who seemed bright, and innovative and who lacked a model for integers which could contribute to his understanding; Robert, who was "correct" but may have had some problems in understanding of integer notions; Jason, who seemed weak in some areas but strong in others, and Connie, who was inconsistent in her thinking.

CHAPTER SEVEN

TEACHING DESIGN

7.0 DURATION OF EXPERIMENT

The teaching experiment consisted of five lessons, over a one-week period (Monday to Friday) from March 9 to 13, 1992, where each student was excused from a regular subject to meet for instruction with the author. After an interval of approximately four days there was a sixth meeting with each individual where a post-test was given in an interview format. Although it would have been more desirable to test the students after a longer period of time had elapsed, this conflicted with a formal examination period at the school.

7.1 LESSON SUMMARIES

LESSON ONE: (45 minutes) Introductory notions of opposites, neutralization, neutral element, equivalence class of neutral element, resultant value, integer notation, equivalence class of resultant value.

LESSON TWO: (30 minutes) Review of neutralization, integer opposites, decomposition, cancellation, resultant value. Addition of chips and integers of the same kind, addition of opposites with chips and integers, addition of chips and integers of unlike sign, mixed additions.

LESSON THREE: (45 minutes) Review of equivalence class of resultant value, additions of all types, cancellation with integers. Subtraction of chips and integers when there is enough to take away; subtraction of chips and integers of equal value; subtraction of chips and integers where adding neutrals is necessary (signs different, and signs same but not enough to take away), mixture of subtractions.

LESSON FOUR: (25 minutes) Review of integer subtraction, notion of set of integers on an extended number line, ordering by position, ordering by magnitude, vocabulary "integer", "absolute value".

LESSON FIVE: (40 minutes) Review of integer addition and subtraction, abstractions involving these operations (reversibility, two ways to achieve one result, size of results).

LESSON SIX: (40 minutes) Post-test on integer notions and procedures

7.2 LESSON OUTLINES AND RESULTS OF INSTRUCTION

7.2.1 <u>LESSON ONE: Introductory Notions</u>

Lesson one was designed to include the first 13 notions from the analysis of understanding based on the mental model, keeping in mind that the notion of neutralization was a powerful idea that had to receive a good amount of attention since the students had not seemed to be so familiar with it. The decision was also made to approach the procedure of cancellation of whole numbers in an expression with the concept of neutralization (subtraction of a number that has already been added will neutralize the addition, and addition of a number that has been subtracted will neutralize the subtraction). It was hoped that this would help the students to cancel both the number <u>as well as</u> the operation that belonged to that number (i.e. the operation preceding it).

The following is the content, methodology, and rationale of the lesson design, which dealt first with the abstract notions of opposites, neutralization, etc., then with the concrete objects of chips and integers.

(1) Discrimination Between Types of Opposites

Since the students seemed to regard the negative positions on the number line as a distinctly different group of positions from the positive or natural ones, with separate rules for ordering, etc., it was necessary to begin to lay the groundwork for the concept that although negative and positive are opposites, they belong to the same set, and the designation of one is only relative to the other. Thus they were presented with two lists of opposites, one group "exact" opposites, and the other "relative" opposites, as follows:

<u>Exact</u>: dead, alive; add, subtract; forwards, backwards; sick, well
<u>Relative</u>: strong, weak; rich, poor; tall, short; hot, cold

The second group also contain the ideas of "neutral" and "arbitrary origin", which are key ideas for the notion of positive and negative as opposites. Discussion centered around how these sets of opposites were different, and focused on the relative opposites with a discussion of how one determines whether something belongs to either of the opposites. This instruction was also a response to the inability of the students to explain clearly how rich and poor were opposites in the pre-instruction interview.

Results of Instruction:

When presented with the two lists of exact and relative opposites, both Jason and Steven were able to see that there was a difference in the two groups of opposites, but were not able to explain their accurate perception that the second group were qualities evaluated on a sliding scale rather than opposing states. Steven was able to volunteer that a person could be rich, but "poor to another person", and after one example, Jason said that hot is "a bit cold". Neither Robert nor Connie could identify any difference in the two lists, and did not volunteer any responses to the explanation given.

(2) Concept of Neutral, and of an Arbitrary Neutral Position.

The students were asked how to determine if someone is weak or strong, and given two scenarios, a student their age among preschoolers, and then among weight-lifters, where they would be strong relative to the first group and weak relative to the second group. The rich - poor dilemma was given in the context of how the instructor would regard herself with respect to a millionaire, and with respect to them. We talked about a spot called "average", and the word "neutral" with respect to the opposites being considered was given at this time. The students were told that the opposites of positive and negative were of interest for the instruction, and asked which group of opposites these belonged to, exact or relative (did they see that there were degrees of positiveness and negativeness?). They were told that the in between position for these opposites was termed "neutral". This notion of neutral was a positional one.

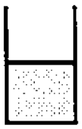
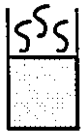
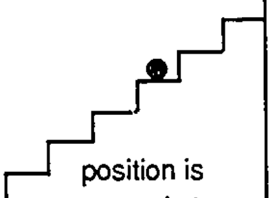
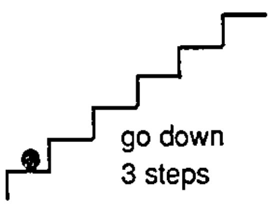
Results of Instruction:

Again, Jason and Steven had no difficulty with the notion of neutrality, and Steven said it was just like in hockey, where the area between the two blue lines was a neutral zone which belonged to neither team. Robert and Connie passively accepted an explanation of the phenomenon when they could supply no explanation of their own.

(3) Concept of Neutralization

Next, the concept of "neutralization" was discussed, in reference to the problem of neutralization of the sour drink which they had done in the pretest. The question was re-read, and the students were asked in more detail what they thought it meant when it said that honey was added to neutralize the taste. This was then described as "undoing" the addition of the lemon by adding an opposite taste, of bringing the taste to a neutral position with respect to the sour and the sweet, a place where neither taste appeared. If they had not mentioned it, it was explained that the honey was not added to make a sweet taste, but only to get rid of the sour taste. They were also reminded of the question in the interview of 647 + 299 - 299, and were asked if this could also be considered as an example of neutralization, and if they could explain how they thought it might relate to neutralization. If they needed assistance, they were told that the action of subtracting 299 neutralized the action of adding 299.

In order to emphasize neutralization as an action, and the arbitrary nature of the neutral position, they were given the following worksheet as a tool for discussion which gave several contexts for neutralization. They were shown the defined neutral state of the first column, an action performed to the neutral state in the second column, and asked "what would neutralize this action?" It was anticipated that for some students, this might be an initial experience with two ways of achieving an outcome, as 3 of the 7 examples could be neutralized in two ways. When they had completed the sheet, they were asked to observe the middle and last columns and to comment on anything they noticed. If they did not see that the actions were opposite ones, using equal quantities, they were directed to this more specifically. They were then told that opposites neutralize each other.

NEUTRALITY	CHANGE (TRANSFORMATION)	NEUTRALIZATION (RETURN TO NEUTRALITY)
position is Montreal	fly north 100 km	
 water at room temperature	 add 10 L hot water	
350 amount is 350	subtract 25 350 - 25 = 325	
 balanced seesaw	 add 60 kg to left side	
(+) (-)	add 3 negatives (+) (-) (-) (-) (-)	
 position is second step	 go down 3 steps	
	flip downwards 	

Results of Instruction:

When discussing the outcome of addition of honey to the sour drink, Robert was the only one who stated that the taste would be "in the middle, in between sour and sweet", which suggests that even though he had been fairly passive in the earlier discussion, it had an impact on his reasoning. The other three students felt that the drink would be "sweetened up", and when reminded of the neutral position, Steven said "so, she was trying to make no taste", but both Jason and Connie thought it would taste both sweet and sour, and needed to be told that in this neutral spot, we say that neither is present, and only the addition of more honey would bring it out of neutrality to make it sweet.

No one experienced any difficulty with the mathematical application of the concept of neutralization to 647 + 299 - 299.

The three boys experienced no difficulty with the tasks on the worksheet which required them to neutralize an action. Jason was the only one who mentioned a second way to approach one of the questions. Connie, however, displayed a lack of knowledge about the physical world during this set of tasks, and this may explain why she was one of the few students who did not solve the temperature subtraction on the pretest correctly. Here, she had no idea of the meaning of "room temperature", and she kept insisting that you could remove the 10 L of hot water from the container (but agreed that she didn't know how to), or that you would have to start all over again by pouring new room temperature water in. When she was told that one could just add cold water, and asked how much could be added, she said to just add some, and remove any extra. All four students said that you would neutralize the addition of 3 negative charges by adding 3 positive ones. Robert and Jason saw the sameness (equal numbers) and the oppositeness (opposing actions) of the neutralization process, and the others saw the opposite nature.

(4) Cancellation as a Neutralization Process

The student was then given a worksheet which contained 4 examples of a string of operations on whole numbers which could be made easier by cancellations. For each, they were asked to use the idea of neutralization to find an easy way to get the answer. They were questioned about each cancellation they performed, and encouraged to focus on cancellation of the operation as well as the number. When they arrived at an answer, they were asked if the answer would be the same if they had performed all of the original operations from left to right. How did they know? Were they convinced? A calculator was used to verify the value of the string, to eliminate arithmetic errors, and to act as an authority. They were given several of these examples in order to overcome the obstacles listed earlier, and to uncover any new ones. Initially, only one example had been planned, but since the first student encountered difficulties, the rest were devised during his session and given to all of the others The following are the tasks used.

(a) $21 + 13 - 7 + 9 + 8 - 13 - 8 + 7 - 21 = \boxed{}$
(b) $18 - 4 + 9 - 12 + 12 - 9 + 17 - 18 + 4 = \boxed{}$
(c) $27 + 22 - 4 + 18 - 22 - 4 - 7 - 18 + 7 = \boxed{}$
(d) $617 - 298 + 321 - 617 + 298 = \boxed{}$

Results of Instruction:

This proved to be the most difficult task encountered in this lesson, for all students.

Before being given any specific instructions, Robert approached the cancellation tasks with a global approach, not the previous left to right strategy evidenced earlier. On the first example, he cancelled from left to right, beginning with the second number, then he addressed the possibility of cancelling the leading number (because it doesn't appear to be added). He felt that it could be cancelled, but wasn't sure how to justify it, except that "you're adding something to 21 here, then you subtract 21 from it". My intervention was to point out that if this question were to be modelled with counters, the initial quantity would have to be counted out, and therefore "added" to an empty surface. This was enough for Robert, and following this discussion, he cancelled, and began all other questions with cancellation from left to right, stating "here you have (leading value), here you're taking it away". He had no other difficulties; as he performed each cancellation, he verbally indicated his steps, and followed each with "so the number (i.e. final result) will be the same after".

In the pre-instruction interview, Jason was the only student who, after being asked to find an easier way to solve, did not see the cancellation in other than the easiest case, and he had shown some sign-related misconceptions. Here, Jason began all cancellations from left to right, and initially crossed out the operation sign following the leading number when asked to draw lines through what was cancelled. After discussion, this problem was not repeated with the leading number, but he asked a couple of times about which sign to cancel (leading or following) with the number which was cancelled. He was able to justify his cancellations with "it would give you zero". Upon completion of the first problem, he felt certain that the answer would not be the same if no cancellations would have been done, and was surprised at the answer when performed on the calculator. He trusted the second and fourth results, but on the third, which had left 27 - 4 - 4, he wanted to verify the equality of the results on the calculator.

Connie cancelled freely, but could not verbalize why this was allowed. She wanted to cancel - 4 - 4 in the third example, but when challenged, was not sure, even though she was aware that they were both subtractions, and had been aware that the others were opposite operations, indicating that she had a stronger focus on the number than on the operations. When she compared results with and without this cancellation, she said "you shouldn't cancel ... because they're the same operation". In the last question, she seemed to be getting tired, and only cancelled the 298's, then asked if she could do the rest on the calculator. Even before she pressed the equal sign, she stated that she could have cancelled the leading number with the subtraction of the same amount.

Steven, who had been the only one to cancel spontaneously in previous tasks, experienced the most difficulty with this procedure here, and all of his problems centered around not being sure which sign belonged to which number, often but not always using the sign following the number. He only made one cancellation in the first example, (21 + and - 21), and had to be encouraged to look for more, but he seemed always to associate the sign following the first number he was considering with that number (and in this way rejected the cancellations of the 13's, since he felt they were both subtractions). He was then asked if subtracting 7 and adding 7 were opposite things, which prompted him to cancel these two numbers. This isolated the 13 problem for him, since he had earlier removed the plus sign preceding the first 13, and this cancellation removed the minus sign following it. Here it was explained to him that the leading sign is connected to the number, but this continued to be a problem in every question he did, despite discussion each time it occurred.

The most common problem for all students was the absence of a sign in front of the leading number, and a desire to then associate the sign following this number with it, which had the effect of causing other similar sign problems within the expression. This was less of a problem if the first cancellations were made within the expression, causing the first number to be isolated from any sign, but still brought comments from all students concerning feelings of insecurity about the ability to cancel that number.

(5) Notion of Equivalence

The third pretest had not yet been given to the grade 7 classes, so each student was presented with the following scenario:

"Let's say you have a large jar of coins here in front of you, with lots of pennies, nickels, dimes and quarters. I'd like to collect 20 cents from you. What coins would you give me?"

After the initial solution, they were asked how many other ways they could find to make 20 cents.

Since part of the concept of equivalence is the interchangeability of representations (Artzt & Newman, 1991), a discussion about whether one representation was preferable over another was held, for example, if everyone gave pennies, that's too much change; if someone was collecting dimes, they would not give up any, etc. Sometimes, in another situation, one might choose a different representation. The important thing was the equivalence of the value, and the fact that it led to interchangeability. The term "equivalent" was introduced.

They were also asked if they knew of any other examples of equivalence, and if not, were asked if they were familiar with equivalent fractions, and were asked to give some examples of this.

Results of Instruction:

None of the students had any difficulty in finding several ways to represent 20 cents, and all knew that they gave equivalent amounts, all interchangeable. Robert could not volunteer another example of equivalence, but the others mentioned equivalent fractions, and all four gave examples of this (all based on one-half). The three boys were aware that although there was a limit to the number of ways to make 20 cents, equivalent fractions could be made in an infinite number of ways. Connie had no notion of infinity, using the term "a lot" of ways.

(6) Introduction of the Chips

Twenty yellow bingo chips each marked with a plus sign, and 20 green bingo chips each marked with a minus sign had been prepared for the instruction. These were shown to the student, and were called positive chips and negative chips. The student was told that they were opposite kinds of chips because they were marked with the opposite signs of positive and negative. One chip from each type was isolated, and the value given to be "one positive" and "one negative", and the student was told that these values could be represented by the use of positive and negative numbers. This was done to give a concrete representation to their previous integer notation. They were requested to mark the positive or negative sign in a raised position so that they would not be confused with addition or subtraction, and in this setting, where no operations were being performed, it was hoped that this would give a stronger association with the raised sign.

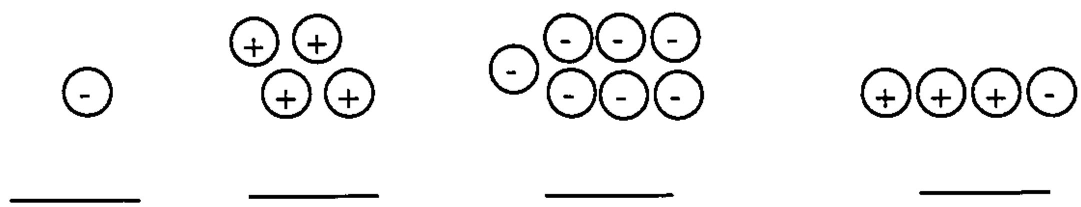
Results of Instruction:

No one had a problem with this representation.

(7) Value of Groups of Chips of the Same Kind, and Integer Notation for this Value.

The student was asked to suggest the value of various groups of chips of the same kind, and to write a number to represent this value. They were also asked to draw a number of plus or minus signs to represent a given integer. They were given the following tasks, where the fourth grouping in the first task contains chips of both kinds, to see if this caused any conflict, or whether the student would spontaneously have a strategy for solving.

WHAT IS THE VALUE OF:



ILLUSTRATE THESE VALUES USING + OR - SIGNS:

(a) +6 (b) -4

In order to see if they would conceive of a meaning for a larger number based on the quantity representation, they were asked to give the meaning of:

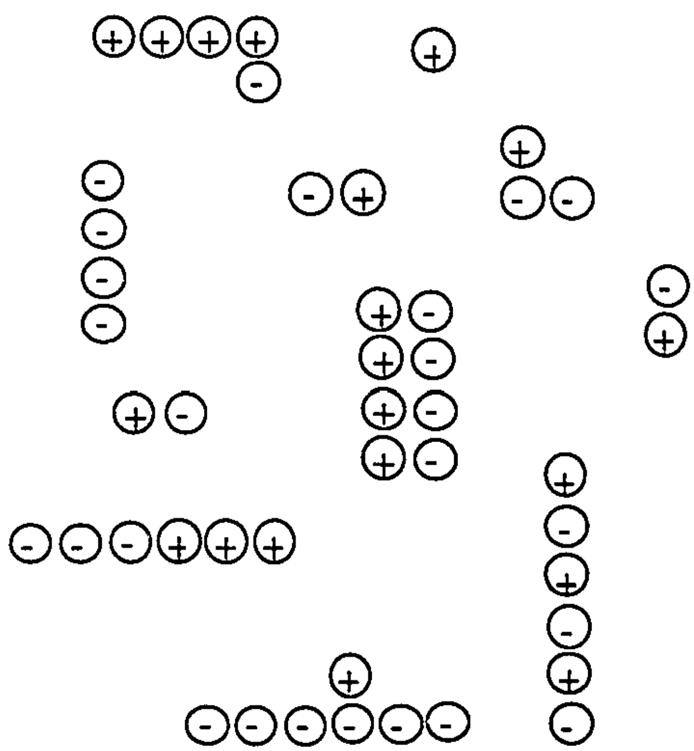
(a) $^{-}713$ (b) $^{+}326$

Results of Instruction:

Jason was the only one who hesitated about the value of a group of chips, and had to be reminded of the value of one chip, then given the value of two chips before he could invent a procedure to determine the value of the group. On the worksheet, when determining the value of 3 positives and one negative, Robert and Steven said they used cancellation to arrive at $^{+}2$, Jason subtracted, and Connie asked "how do you know if you're adding or subtracting?" and was not able to solve when told that there was no operation involved. Robert, Steven and Connie included "713 green chips" or "713 negative signs" as a representation of $^-713$, while Jason included "minus 713" and was reminded that when the sign is raised, it does not represent an operation.

(8) Neutralization of a Positive and a Negative Chip, Creation of a Neutral Element, and Equivalence Class of Neutral Element and Neutralization

The student was asked what would happen if there were one positive and one negative chip. If they did not volunteer the term "neutralization", it was supplied to them. They were then asked if they could make their own combination that would neutralize, then to make another. Given a certain number of chips of one kind, they were asked to neutralize them. They were asked how many ways they could think of where neutralization could occur in this context. They were given the following worksheet, where they were to circle the neutral elements that they could find.



Results of Instruction:

Connie was the only student who had to be told that a combination of one positive and one negative was an example of neutralization, the others spontaneously had this notion. Everyone could supply more examples of neutralization, and the neutral element. Before any vocabulary was introduced, Robert termed the combination "neutrals" and Steven used "zero". When circling neutral elements on the worksheet, all 4 students circled groups rather than 1-1 combinations, and Jason was the only one who also isolated the neutral elements in the uneven groupings. Steven circled a single + with a neighboring group of + - -. Thus the concept of neutralization was an intuitive one for three of the students, and one easily adopted by the fourth. There were no problems with the equivalence of various representations of the neutral element.

(9) Opposites of Integers

Since a certain number of negative chips could be neutralized by an equal number of positive chips, the student was told, if they had not already mentioned it, that these values can be seen as opposites, and then they were asked to give opposites to specific integers, for both one digit and larger integers.

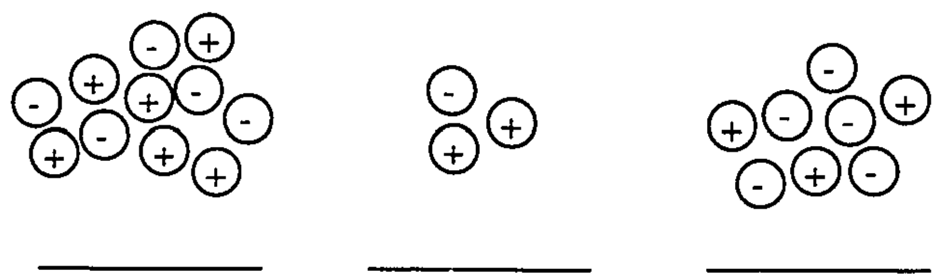
Results of Instruction:

No one had any difficulty with this notion, Steven in particular basing it on neutralization.

(10) Resultant Value of a Mixture of Different Kinds of Chips, and Integer Notation for such

The student was presented with a group of chips which contained both kinds of signs, and was asked to give the value of the group, to see if they would be able to come up with the procedure of making neutral elements, and using the residue. If they did not do so, they were shown how to. Different groups were given and they were requested to write the value in integer notation. It was thought that perhaps there might be confusion about using a notation that had previously represented only one kind of chip, so in this eventuality, it would be dealt with in the context of equivalence. They were given the following task.

WHAT IS THE VALUE OF:



They were also verbally asked "If I had 80 negative chips and 20 positive chips, what value would I have?" to see if they could use the model mentally to determine resultant value.

Results of Instruction:

The three boys all invented a way to determine the correct value of a mixed group, Steven cancelling the equal amounts, Robert "neutralizing", and Jason adding each group, then performing subtraction. When Jason was shown that it would be easier to match the opposites, he adopted this procedure. Connie, however, could not apply the concept of neutralization without assistance, but took over the matching process before the instructor had finished, and used it for all other examples.

When asked the value of 80 negative and 20 positive chips, Robert stated that "the 20 positive neutralizes 20 of the negatives, and there are 60 negatives left over". Steven said that he subtracted because "it's opposite, you're going to add it, then subtract it", and when pressed for a further explanation, he used the concept of the number line, and the rule of doing the opposite operation, then he said "or I could have just equalled them up", so even though he hadn't used the neutralization notion, he indicated that it <u>was</u> available as a mental model for him. Jason also said that adding 20 was like subtracting 20, but could provide no explanation for it. The situation was changed to 8 negative and 2 positive, and asked how to find the value, and again he subtracted, so the chips were put out, and when asked what was subtracted, he was able to state that the matching ones were subtracted out. Thus for Jason, this procedure was not a mental model for larger numbers.

(11) Equivalence Class of Resultant Values

The student was asked to put out chips to represent a certain integer, then asked if there was any other way that integer could be represented. If they did not come up with the addition of neutrals on their own, a unit neutral element was added, and the value of the group was asked for. They were then asked to make another representation, etc. They were asked how many ways they could think of to represent the given integer. They were then given the integers $^{-}4$ and $^{+}1$, and were asked to draw three equivalent representations for each.

Results of Instruction:

When asked to make three equivalent representations for $^{-}4$ and for $^{+}1$, the three boys had no difficulty, and stated that there are an infinite number of ways of representation for each value. Connie used only two representations, symbols of one kind, or symbols with a neutral element added, and when pressed for more, this distinction was made more clear as she asked "adding more chips?". Once that was clarified, she had no problem with the notion of "a lot of ways" that each could be represented.

INDIVIDUAL STUDENT PROGRESS IN LESSON ONE

Steven was very actively involved in the lesson, making associations, suggesting examples from his own life, seeing ahead and drawing conclusions. His only difficulty was with the cancellation exercises, due to his ambiguity about which sign should be considered for a particular number. This may not have been completely cleared up for him during the instruction.

Robert had no difficulty with any of the notions or procedures, except for exhibiting the temptation in the cancellation exercise to associate the operation (addition) after the leading number (in an expression) with that number, which was quickly and easily cleared up.

Jason's perception of cancellation seems to have benefited from the notion of neutralization, as he showed great improvement in awareness of the possibility of cancellation, and in his linking of numbers with operation signs. He did not spontaneously associate groups of chips of one kind with integers, as the others had, and when given a situation with numbers that had not yet been modelled, his orientation was to perform subtraction even when an operation was not indicated. He could correctly perform procedures with the chips, and write an integer result, but did not seem to be able to use these procedures when given an integer context. He was the only student who did not adopt the convention of raising the sign of the number.

Connie's style of response was often imprecise, and she had to be questioned more directly at times to elucidate a more detailed answer (for example, when asked how she would give 20 cents, she said, "I'd give you nickels, dimes and pennies, all mixed up, some of everything"). Sometimes she was unable to verbalize her thinking. Once shown a procedure, Connie was able to understand how it worked, and adopt it, but she was unable to apply it in a different context, or to use it to construct understanding for it in a broader context. For example, she was told that one positive chip and one negative chip combined with each other was worth "nothing" due to neutralization, but when asked for the value of 3 positive and 3 negative chips, her answer was "3 positive and 3 negative". After identifying neutral elements in groups of chips, with the knowledge that they had a value of "nothing", she could not apply this knowledge, without intervention, to find the value of the entire group. Although Connie needed more intervention than the others did regarding the neutralization model, the instruction was not difficult for her to understand, and she had no difficulty with any of the procedures.

EVALUATION OF LESSON CONTENT

The neutralization model provided the students with the notion of the value of an integer, and with the concept of neutralization. It supplied them with another example of equivalence classes, and with these basic elements of understanding, the other concepts of neutral element, opposites and resultant value were often spontaneously constructed or easily accommodated by all of the students.

7.2.2 LESSON TWO: Addition

Lesson Two was designed to deal with the operation of integer addition, modelling this first with the chips. The lesson began with testing of the notions and procedures that were necessary for addition. The following notions and procedures formed the content of the lesson.

(1) Formal Testing of Previous Notions

(a) Neutralization: The students were asked to give an example of neutralization, to ensure that the concept was understood.
Results: All four students gave good examples of neutralization, two from real life (Connie and Steven) and two with integer connotations.

(b) Opposites of Integers: The students were asked to give the opposite of negative 34, as the concept of the addition of opposites would be dealt with in this lesson.
Results: All four correctly identified the opposite of a specific integer, two giving number line references (Jason and Steven) and two giving the oppositeness of positive and negative.

(c) Decomposition: Since this procedure is needed for the addition of integers of unlike signs, two examples were given to the students, as follows, the first given the leading number, and the second given the second part:

(a) $185 = 130 + \boxed{}$

(b) $215 = \boxed{} + 200$

Results: All four students used subtraction to find the missing number, and were convinced of their answer (some checked by addition).

(d) Cancellation: Since there had been so much difficulty with the procedure of cancellation, the following task was given:

$$87 - 14 + 32 + 14 - 29 - 87 + 29 = \boxed{}$$

Results: None of the students had any difficulty or hesitation with any of the cancelling in this task.

(e) Resultant Value: Since this procedure was necessary for the determination of the value of a group after addition, this was tested for a group of chips of the same sign, a group of different signs, and a description of a group with larger quantities, as follows:

Write the value of: a) - - - - - ________

b) + + - + + - - - + + + + - __________

c) a group of chips that has
89 positives and 17 negatives. __________

Results: No one had difficulty with the first task. On the second task, Steven and Robert counted each kind of symbol and subtracted, Connie crossed out matching pairs from left to right, and Jason said that he had forgotten how to find the answer, but when it was brought to his attention that it was a combination of positives and negatives, he asked if he was to match them. He did so in a disorganized fashion mentally, and got the wrong count, so the instructor drew boxes around each pair as he re-did the exercise.

On the third task, all four students subtracted to get the right answer. When asked for the reason for subtraction, Jason stated "it cancels matching up", Robert said "the 17 negatives neutralizes 17 of the positives", Connie said you would "take out 17 from 89 positives", and Steven said that if you take out the positives, "now it's like zero". All four students seem to be using the concept of neutralization of the matching amounts.

(2) ADDITION OF LIKE SIGN (CHIPS AND INTEGERS)

(a) CHIPS The students were given two groups of chips of the same sign, and asked to add them, and give the result. If there was any difficulty with the task, the procedure of grouping them and counting would be given. They were then asked to write number sentences to match the chip additions.

(b) MATHEMATICAL ADDITION OF INTEGERS OF LIKE SIGN (TRIVIAL ADDITION)

The following integer addition problems were presented to the students to see if they could use the chip procedure as a mental model:

(a) $^{-}8 + {}^{-}12 =$ (b) $^{+}6 + {}^{+}5 =$

(c) $^{+}100 + {}^{+}165 =$ (d) $^{-}215 + {}^{-}32 =$

Results of Instruction:

No one had any difficulty with the addition of chips or integers of the same sign.

(3) ADDITION OF EQUAL AMOUNTS OF CHIPS OF OPPOSITE SIGN, AND ZERO AS NEUTRAL ELEMENT; ADDITION OF INTEGER OPPOSITES

The students were given two groups of chips, equal in number but opposite in sign, and were asked to add them together, to give the resultant value, and to write the integer sentence for the addition.

Following this, the students were given the following pencil and paper problems, the last two being such that "cancellation" due to neutralization could be considered.

(a) $^{-}825 + \boxed{} = 0$

(b) $\boxed{} + {}^{-}29 = 0$

(c) $^{+}3145 + {}^{-}3145 = \boxed{}$

(d) $^{-}25 + 0 = \boxed{}$

(e) $^{-}49 + {}^{+}49 + {}^{+}32 = \boxed{}$

(f) $^{-}86 + {}^{+}21 + {}^{-}21 = \boxed{}$

Results of Instruction:

None of the students had any difficulty with the addition of opposite chips or integers, presumably as a result of working with the neutral element, since both Steven and Robert used the word "neutralize" to justify the result, and Jason said you had to use "the exact opposites" to get zero. Connie stated that this result would happen if you have the same numbers, one positive and the other negative. All of the students saw that this result would happen in an infinite number of ways. Identification of zero with the neutral element was spontaneous for all of the students, Connie first calling it "nothing" and writing "0" as the result of the operation.

Missing in these tasks was the opportunity to neutralize when the opposites are not consecutively placed, as might happen in writing a number sentence for a chip manipulation with the addition of "zeros". For example in $^{+}9 + {}^{-}6 + {}^{+}2 + {}^{-}9 + {}^{+}7$.

(4) ADDITION OF CHIPS AND INTEGERS OF OPPOSITE SIGN

The students were given two groups of chips of differing amounts and of opposite sign, and were asked to add them, and to write the integer sentence that described the addition.

The students were then given the following tasks, and if they experienced difficulty in finding the answer, they were directed to model it with the chips for the first four examples. Any student still having difficulty was given more chip exercises, and more tasks with small numbers. Any student who was unable to use the chip addition as a mental model for the larger numbers was shown how to write the decomposition and cancel the numbers which neutralize. The others were not shown this written step, as it was intended only as a bridge between the chips and the mental model.

(a)	$^{+}5 + {}^{-}2 =$	(b)	$^{-}4 + {}^{+}18 =$
(c)	$^{+}9 + {}^{-}15 =$	(d)	$^{-}11 + {}^{+}6 =$
(e)	$^{-}66 + {}^{+}50 =$	(f)	$^{+}54 + {}^{-}100 =$
(g)	$^{+}215 + {}^{-}5 =$	(h)	$^{-}111 + {}^{+}130 =$

Results of Instruction:

Steven was able to solve these problems by neutralizing mentally. He spoke of trying to add on to the smaller number to come up with the unneutralized amount (for example he added 46 on to 54 to get 100 in $^{+}54 + {}^{-}100$), but in one example ($^{-}66 + {}^{+}50$) he said "I'll just do it the way I usually do, I'll just subtract it". His comments in the interview, "I change my additions to subtractions", and "you <u>can</u> add, but I don't know how to" must be re-evaluated in this light to take on the meaning of: I use the procedure of subtraction because I don't know an <u>addition procedure</u> that will solve the problem. Here his addition procedure was to <u>add on</u> from the "smaller" number, then check by addition to see if the two numbers added up to the "larger" amount. Steven was shown how to write the decomposition for the additions, but it did not seem to make sense to him, as he often wrote the "larger" number rather than decomposing it, or tried to add zero, etc. He seemed to have no need for this intermediary step, since he was able to mentally solve the additions, and in fact he said the writing of these sentences confused him. He knew that you could use the sign of the number that represented <u>more</u> of a certain kind.

Robert had no difficulty with any of the additions, for each stating, (for example) "because the negative 7 neutralizes 7 positives". He did not state whether he subtracted or added on, but he got all answers very quickly, so decomposition was not a problem. He could write the mathematical decomposition, but he did not need it, and it was just an exercise for him.

Jason started out hesitantly, but when his thinking was consolidated by the instructor's mention of "making neutrals", he was able to solve all chip problems mentally. When the move was made to paper and pencil exercises, he used the chips to solve the first one ($^{+}5 + {}^{-}2$) and did the second ($^{-}4 + {}^{+}18$) by subtracting (in the pre-instruction interview, he had been able to solve mixed additions as long as the negative number came first, as it does in the second task). For the third ($^{+}9 + {}^{-}15$), he was unsure whether he should subtract or not, so he was reminded about matching chips, and he was able to solve it, then he asked, "Would it always be the right answer if you did the bigger number subtract the smaller one?", as he sought to find a rule that would help him solve any addition. Jason was also shown how to make the decomposition sentences, but each time he needed intervention to build them.

Connie could do the additions with the chips, but made a couple of errors when first solving mentally (subtraction and sign errors), then was successful on all others. She stated that she used a rule (based on the concept of neutralization) that the sign of the "bigger" number "would be the answer" because there would be no more of the "smaller" number, and you subtract the smaller amount out of the larger group. Connie was given this lesson a day later than the boys, and since the decomposition sentence had caused unnecessary confusion for Steven and Jason, this technique was not shown to Connie, since she had no need for it.

All of the children were able to understand this type of addition, in the physical environment of the chips and in the mental environment of integers, as combining two groups of opposite types where neutralization of matching quantities would yield the resultant value. The decomposition sentences seemed to be an unnecessary obstacle centered on mastering an arithmetic procedure, rather than being understood as a mathematical representation of a physical manipulation.

(5) REVIEW OF ALL TYPES OF ADDITIONS

In order to see if the student could now sort out the different types of addition and succeed at solving them, the following questions were given as paper and pencil tasks. If difficulties were encountered, the student was encouraged to use the chips to solve.

(a) $^{+}5 + {}^{+}11 =$	(b) $^{-}6 + {}^{+}6 =$
(c) $^{-}19 + {}^{+}4 =$	(d) $^{-}3 + {}^{-}8 =$
(e) $^{-}3 + {}^{-}6 + {}^{-}4 =$	(f) $^{+}18 + {}^{+}9 + {}^{-}9 =$
(g) $^{-}9 + {}^{+}2 + {}^{-}6 =$	

Results of Instruction:

The mixed additions caused no problems, except that for $^{-}3 + {}^{-}8$, Connie focused on the minus signs and subtracted, and needed intervention before she added. The task $^{+}18 + {}^{+}9 + {}^{-}9$ was the first time they had met the possibility of cancellation with integers. Steven and Jason spontaneously cancelled, while Connie and Robert solved from left to right, then said they could also have cancelled.

(6) RULES FOR ADDITIONS

All four students were able to verbalize that when the signs are the same, you use that sign, and add the two numbers. When the numbers are opposites, there is neutralization, or "cancelling out", and the result is always zero. When the signs are different, the sign of the larger number is used because there are "more" of that kind, and you subtract because a quantity equal to the smaller number will match up.

INDIVIDUAL STUDENT PROGRESS IN LESSON TWO

All four students were able to construct their own procedures for integer additions based on the neutralization model, Connie and

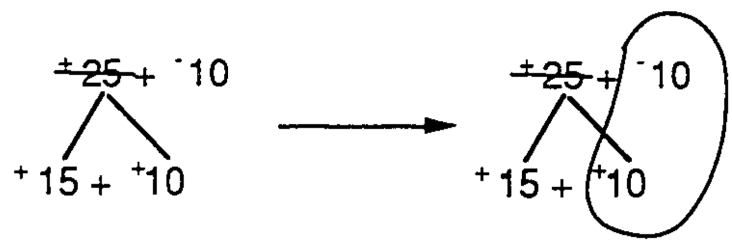
Jason formalizing rules for each type without being asked to. Steven and Jason verbally thought each step through, even for tasks which were similar, asking questions, checking, etc., while Robert always used the same vocabulary for each task, word for word. This is in contrast in particular with Steven, who for the same type of task, used a different phrase each time ("I neutralized", "I just took two of these", "you put 9 aside", "you add, you put 6 together"). It seems that Robert's understanding may be a rote understanding, while the others seem to be reacting to each new question. Connie was successful only when she was aware of all of the elements of the task, but there were times when she focused on one element only (for example the sign of the operation, disregarding the signs of the numbers), and made errors, but did not notice the conflict. All students used raised notation for integers, but Steven was the only one who used it consistently.

EVALUATION OF LESSON CONTENT

All four students clearly understood (a) that when adding chips or integers with the same sign, you end up with more of that kind (b) the addition of opposites neutralize to result in zero, and (c) neutralization yields the results of addition. In all cases, the physical model provided a basis for the child to construct a corresponding mental model for integers.

Addition of integers using the neutralization model was not a problem for any of these students, and none of the notions conflicted with previous knowledge or intuition. It seems that the quantity representation now puts an emphasis on whether the signs are the same or different, and eliminates the problem that most of the students had of adding on a negative number.

Integer additions did not need the intermediate step of the "mathematization" by decomposition, which only confused the students when introduced. Perhaps it also did not exactly match the chip situation, and became a manipulation of numbers rather than a way of representing what happened physically. The chips were not really "decomposed", but matched, so perhaps a diagram (such as the one below showing a physical match as opposed to a mathematical match) rather than a sentence would be effective if needed.



The issue of vocabulary emerged with both Jason and Steven, and the concept of "addition" was at issue. As discussed earlier, it was seen that Steven makes a real distinction between addition and subtraction, and feels that he is not adding when he uses subtraction to solve the addition of unlike signs. When Jason was first given two groups of chips of unlike kinds, and asked to add them, he said "I wouldn't really add them, I would count them." The set of integers is the first environment in which the notion of "addition" is broadened to include procedures other than totalling a number of groups.

7.2.3 LESSON THREE: Subtraction

Lesson three was designed to teach integer subtraction, and this was viewed as the critical lesson, since subtraction has been the operation to cause the most difficulties in the domain of integer arithmetic. The lesson began with a bit of an addition review, as well as a task requiring the notion of equivalence classes of resultant values, as the procedure of adding neutrals to the leading number in a subtraction when there is not enough to take away depends on the assurance that the value of the leading number will not be changed.

(1) Formal Testing of Previous Learning

(a) Notion of Equivalence Class of an Integer: The students were asked to perform the following task, and any who had difficulty would be shown different configurations of chips that would all result in the same value, then be given the task again.

Show 4 ways to represent $^{+}5$

Results: Connie, Steven and Robert easily gave 4 equivalent representations of $^{+}5$, but Jason gave two only, one without neutral elements, and one with, and he stated that he didn't know any other way. When he was asked to show something else that was equivalent to $^{+}5$ with the chips, he said "something else just like this?" and he put out 5 positive chips, and it took more discussion before he added any other representation of the neutral element to the chips.

(b) Cancellation of Integers: The following task was given to see if cancellation would be performed when the addition of opposites was separated by another number.

$$^{+}18 + {}^{-}43 + {}^{-}18 + {}^{+}13 = \_\_\_\_\_$$

Results: Steven and Connie cancelled correctly, with no sign problems, and Steven went on to solve correctly, but Connie added 13 and 43, perhaps ignoring the negative sign and focusing on the addition sign. Jason also cancelled, but asked about the addition sign after the +18: "this would go out too, eh?". Robert was back into his left-to-right solution procedure, and noticed the cancellation only when he was at the addition of $^{-}18$, and he also showed that he would cancel the addition sign following the +18 and use the addition sign following the $^{-}43$ in the cancelled version.

(c) Additions of all types: Large numbers were used first to see if students had retained the notion of integer addition, but if the large numbers proved to be difficult, examples with one digit numbers were substituted in their place (made from the tens digit of the original number i.e. $^{+}21 + {}^{-}81$ became $^{+}2 + {}^{-}8$). The following tasks were given:

(a) $^{-}8 + {}^{-}8 =$ (b) $^{+}21 + {}^{-}81 =$

(c) $^{-}48 + {}^{-}16 =$ (d) $^{-}49 + \square = 0$ (e) $^{+}3 + {}^{-}8 =$

Results: Robert, Steven and Connie had no difficulties with these additions, and spoke of "neutralizing" or "making zero" for the additions with both positive and negative integers. Both boys forgot to write a negative sign, although it had been there for them verbally. Jason could not remember how to add when the signs were mixed and the first number was positive. Simpler numbers were given and he solved immediately, using the concept that the first amount would be "lowered".

(2) USE OF THE CALCULATOR TO VERIFY INTEGER OPERATIONS

The calculator had always been available to the students, and had sometimes been used by them to verify results of whole number operations. Here it was introduced as a tool for integer operations, and was used to check the above answers to develop confidence in its ability to handle integers. It would be available from this point on to check any answer which a student was uncertain of even when the chips had been used (to validate the chip operations as well as the integer operations).

Results of Instruction:

All three boys showed surprise when told that they could use a calculator to verify their results:

Jason: "can you do negatives on it?"

Robert: "do we have negative?"

Steven: "you mean you can do what I did there?"

Steven and Robert expected to use it in the same way in which they had done their own operations (for example, for $^{+}2 + {}^{-}8$, they were prepared to do 8 minus 2 and somehow get the right sign - this would be surprising!). Once shown how to use the keys correctly, Robert was the only student who had any difficulty procedurally, as he thought the +/- key would also be used for the operation, but this misconception was quickly dispelled. All students showed trust in the calculator results.

(3) SUBTRACTION OF CHIPS AND OF INTEGERS OF THE SAME KIND WHEN THERE ARE ENOUGH TO TAKE AWAY (TRIVIAL SUBTRACTION)

On the same page with the above additions, the last task was the following: $^{-}10 - {}^{-}3 = \square$. This was positioned to create a bit of a confrontation with a "trivial" subtraction, to see whether the student would be able to make sense of such a subtraction. Unfortunately this type of subtraction was not pretested (with the exception of $^{-}5 - {}^{-}5$ as an interview task), so it is not known whether they could have solved this using strategies outside of the neutralization model. If the student had no difficulty in obtaining the correct answer in this example, they were asked to explain their reasoning, and then were given the following four examples. If there was any difficulty, this question was modelled with the chips, and then the following tasks were given.

(a) $^{-}9 - {}^{-}5 =$ (b) $^{+}21 - {}^{+}11 =$

(c) $^{+}972 - {}^{+}222 =$ (d) $^{-}82 - {}^{-}14 =$

Results of Instruction:

None of the students needed to use the chips to model this subtraction, and no one experienced any difficulty, treating the tasks as "normal" subtractions. They appear to have extended the quantity concepts in N to tasks totally in the integer domain (i.e. both numbers negative).

(4) SUBTRACTION OF INTEGERS OF EQUAL VALUE

For the task $^{-}5 - {}^{-}5$ in the interview, Jason and Robert had described it as a "regular" subtraction, and gave zero as the result. Connie had also subtracted, but put negative zero as the answer, since the two numbers were negative, and Steven initially put zero, but changed it to $^{-}10$ when he tried to do a sign change. It was anticipated that there would be no difficulty with this concept at this stage, but the following two questions were asked just to make the point that zero is a result of <u>subtraction</u> now, not of <u>neutralization</u>. If there was any difficulty, a similar task with chips would be given.

(a) $^{+}67 - {}^{+}67 =$ (b) $^{-}812 - {}^{-}812 =$

Results of Instruction:

These were also treated as "trivial" subtractions by all students.

(5) CANCELLATION INVOLVING SUBTRACTION OF INTEGERS

In order to prepare the student for the writing of expanded sentences for integer subtractions (e.g. $^{+}6 - {}^{-}9$) where the addition of neutrals would be shown ($^{+}6 + \boxed{^{-}9 + {}^{+}9} - {}^{-}9$), the following task was given, where one of two cancellations is possible (either $+ {}^{-}9 - {}^{-}9$ or $+ {}^{-}9 + {}^{+}9$), and one that may appear to be possible ($+ {}^{+}9 - {}^{-}9$) is not.

$$^{+}6 + {}^{-}9 + {}^{+}9 - {}^{-}9 = \_\_\_\_\_$$

After the student had made one cancellation, before the answer was checked, the sentence was re-written by the instructor, and the student was told that another way of cancellation was possible and was asked if he could find it. This would guarantee that at least one cancellation was a correct one (although it would not guarantee correct use of operation signs after cancellation), and would point out that there are <u>options</u> for cancellation with integers. It was expected that one result would likely be right ($^{+}6 + {}^{+}9 = {}^{+}15$), and one would likely be wrong ($^{+}6 + {}^{-}9 = {}^{-}3$ as the result of the incorrect cancellation, and $^{+}6 - {}^{-}9 = ?$ as a situation which they have not yet encountered). Discussion about the expectation of the student to receive the same answer from different cancellations would be held. The calculator would be used to evaluate the string from left to right with no cancellations, and answers would be compared. The source of incorrect answers would be confronted (incorrect cancellation, or incorrect subtraction), and this "new" kind of subtraction would be approached head-on.

In retrospect, it seems that this one example should not have been counted on to suffice here, as the impact of this confrontation should have been tested by other examples that would force the student to evaluate the criteria for cancellation, and examples would have to be carefully chosen so that subtraction types that they had not yet encountered would not result. In a sense, the above task was also designed to lead into the need for a procedure to deal with these subtractions, so this could have been the last of a group of cancellation tasks. A better task to begin with would have been $^{-}6 + ^{+}2 + ^{-}1 - ^{-}6 =$ ___ .

Results of Instruction:

All but Jason cancelled the addition of $^{-}9$ with the addition of $^{+}9$, but Robert and Steven both wanted to jump off with the plus sign following the $^{+}6$ after this cancellation. All four students performed the incorrect cancellation of the addition of $^{+}9$ and the subtraction of $^{-}9$, and when challenged on it, all admitted to either concentrating only on the sign of the number, or on the sign of the operation, but not both. The cancellation of the addition of $^{-}9$ with the subtraction of $^{-}9$ was seen last by all of the students (when asked to still find a way to cancel). No one except for Robert seemed bothered by the difference in solutions obtained by the different cancellations.

(6) SUBTRACTIONS OF CHIPS AND INTEGERS REQUIRING ADDITION OF NEUTRALS DUE TO AN INSUFFICIENT AMOUNT FOR SUBTRACTION

(a) CHIPS OF THE SAME SIGN

The problem of how to take away what's not there was presented in a physical setting, and the students were asked if they could think of any way that the subtraction could be performed. If they did not see any way, they were then asked if there was any way that they could represent the leading number so that there would be enough to subtract. If these questions did not produce the desired strategy, the student would be shown the addition of one neutral, and be asked if the value of the number had changed, and if it was alright to do this, and neutrals would continue to be added by the instructor in a counting-on manner (now we have 6 positives, etc.) until the student took over. Examples were given using the chips until the student seemed secure in the procedure.

Results of Instruction:

All of the students recognized the conflict of being asked to subtract when there were not enough to take away. When presented with the first task, the replies were:

J: (2 positives subtract 11 positives) "Can't"

C: (2 positives subtract 8 positives) "Can't, cause there's only two"

S: (3 positives subtract 8 positives) "How am I supposed to subtract positive 8? You need the other 5 chips. So this is the wrong way."

R: (6 positives subtract 10 positives) "Well, the 6 are gone, and there are supposed to be negative 4 ... cause it's 4 below zero."

None of the students had any difficulty with the concept of change in representation of the first number, but both Jason and Connie felt that you still couldn't subtract because you had the original number and zero (the changed representation was not yet viewed as a unit, but as two sets). When the instructor physically mixed them together and verified that they were convinced that the group still had the same value as the original number, both of them subtracted willingly, and Connie made the "mixing" a step in her subtraction procedure.

For the most part, the procedure of adding the neutrals to the initial group was performed correctly, although each student very occasionally added only chips of one kind. Only Jason added one neutral at a time, while the others added the group they needed, then the group opposite to that. The determination of how many neutrals to add took various forms. Connie added at least the full amount to be subtracted, and just neutralized any extras after the subtraction. Steven usually added as many neutrals as the subtrahend, and Jason also did this once. Robert may have not had enough examples here, as he needed help with the procedure each time.

Steven was the only student who initially did not have confidence that the manipulations with the chips gave the "right" answer, and it seemed to be an obstacle that the result was of a different sign than both of the original integers.

Instructor: What's the answer?

Steven: "negative 5. At least that's what it comes out to"

and later,
Steven: ($^{-}4$ - $^{-}6$) "and you're left with positive 2. But we're not doing that, cause these weren't positives."

Verification of the answers on the calculator helped him to accept the results of the concrete subtractions as valid (he had trusted in abstract rules with no model to back them up before the teaching experiment had begun, and he had been unable to perform subtractions using his rules). This freed him to begin to reason about the results. Connie and Jason both trusted chip results when they differed from results obtained by their own reasoning strategies.

(b) INTEGERS OF THE SAME SIGN

The following paper and pencil tasks were given, where the students were allowed to use the chips to solve if needed, and they could verify their answers with the calculator if necessary. If they were using the model mentally, yet were unable to keep track of what was happening, the extended sentence (e.g. $^{+}3 + \boxed{^{-}4 + {}^{+}4} - {}^{+}7$) was introduced to show the addition of the neutral elements, and grouping and cancellation showed the desired subtraction, as follows:

$$^{+}3 + {}^{-}4 + {}^{+}4 - {}^{+}7 \quad = \quad {}^{+}7 + {}^{-}4 - {}^{+}7 \quad = \cancel{^{+}7} + {}^{-}4 - \cancel{^{+}7} \quad = {}^{-}4$$

(a) $^{+}3 - {}^{+}7 =$ (b) $^{-}8 - {}^{-}10 =$

(c) $^{-}1 - {}^{-}5 =$ (d) $^{+}3 - {}^{+}11 =$

Results of Instruction: Both Steven and Robert used the extended sentences, Steven finding them "easier" than using the chips (perhaps because it became a manipulation he could concentrate on, rather than trying to reason about the subtraction). He always set the sentences up so that the first 2 numbers were of the same sign and combined to cancelled with the subtraction, leaving the third number as the answer, as follows:

$$^{+}3 + {}^{+}4 + {}^{-}4 - {}^{+}7 \quad = \quad \cancel{^{+}7} + {}^{-}4 - \cancel{^{+}7} \quad = \quad {}^{-}4$$

It seemed that the notion of neutralization had given Jason reasoning that he could transfer to the number line situation, which had been his mental model before the instruction. Zero had previously been a significant position (for example, to subtract a larger natural number from the smaller, he subtracted down to zero, then the rest went over to the negatives, a technique which Peled et al (1989) describe as indicating use of a mental model of a divided number line partitioned at zero). It may be that zero as the result of neutralization fit in with this notion and strengthened it (group to make zero, then take the left over), as now he was able to subtract with negative numbers, for example:

Jason: ($^-8$ - $^-10$) "smaller negative, and you're subtracting the big negative, so it's gonna have to go above zero, because there won't be place for the 2 left to be below zero".

Connie solved these all correctly, very quickly, using a rule that when "you're minusing two positives, it will give you a negative answer, and when you're minusing two negatives, it will give you a positive answer", and she just subtracted the "smaller" number from the "larger" one.

Neither Connie nor Jason were shown the writing of the expanded number sentence, since Jason was not using the chips as his model, and since Connie had added as many neutrals as she felt like, even after intervention to point out the minimum necessary. The use of the expanded sentence would have complicated the numerical task for her. A task such as $^+8$ - $^+17$ could become $^+8$ + $^-17$ + $^+17$ - $^+17$ if the number of neutrals added was equal to the number to be subtracted, leaving the addition of opposites rather than the addition of like signs. An even more complicated situation could develop if a random quantity of neutrals were added, which often happened with the chips, for example $^+8$ + $^-25$ + $^+25$ - $^+17$ where the subtraction and the cancellation of the extra neutrals would be well hidden and would defeat the intended simplicity of modelling the chip procedure with numbers.

(c) CHIPS AND INTEGERS OF OPPOSITE SIGNS

The next written task was of this new type, and the students were given this as a paper and pencil task to see if they would notice the similar conditions (not enough to subtract). If assistance was needed, the chips were used to model the subtraction.

(a) $^{-}9 - {}^{+}2 =$	(b) $0 - {}^{+}9 =$	(c) $^{+}4 - {}^{-}6 =$
(d) $^{-}2 - {}^{+}8 =$	(e) $^{+}12 - {}^{-}3 =$	(f) $^{+}8 - {}^{-}8 =$
(g) $^{-}7 - {}^{+}5 =$	(h) $0 - {}^{-}7 =$	

Results: Steven had the least trouble with these subtraction tasks in the context of the neutralization model, needing little intervention from the instructor. He felt that writing the expanded sentence was easier than using the chips or trying to reason, and he used this procedure for almost all of the tasks. While he could usually write the sentence correctly, he often cancelled either the two neutrals added, or an illegal combination, but when challenged, he was able to recognize that he had not been concentrating on the signs, or on the task (subtraction), and thus was able to correct himself. He attempted four of the tasks mentally, but without referring to the neutralization model. All were incorrect, and he used the expanded sentence to get the correct answer. His approach was procedural rather than based on understanding of the subtractions.

Connie tried to make rules from previous examples and from chip results, and for the most part was able to discriminate which type of problem was being addressed. Almost all of her rules were based on the signs of the numbers, as when the signs were different, she said "you <u>can't</u> subtract (physically), so you add (the two numbers)", and she had noticed that the answer would have the sign of the first number. For the tasks that involved subtracting from zero, she used the criteria that it had to be the addition of opposites that would give zero. If an answer derived from a rule differed from a chip manipulation answer, she trusted the chips, but never used them on her own, only when asked to do so.

Jason also did most of the tasks within a mental framework, referring to number line movements. The only type that he was not able to reason out were the ones that had zero as the minuend. For these, he used the chips and had confidence in the answer.

Robert had a lot of difficulties with the subtractions. He needed help for most of the tasks, whether he chose to use the chips or the expanded number sentences to solve with. He was often confused about placing the numbers when writing the expanded sentences and would forget to add in the neutrals, and then he would often want to cancel the neutrals added. He also was not always sure of how many neutrals to add in. His weaknesses here seemed to stem from not being able to subtract consistently with the chips.

(7) MIXED SUBTRACTIONS

Subtractions with two- and three-digit numbers were given to force the students to a position where they would either use the chip procedures as a mental model, or where they would have to use the expanded sentences in order to solve the following problems. If there were difficulties in abstraction, the student was given a similar one-digit problem to solve with the chips first.

(a) $^{+}100$ - $^{+}145$ = (b) $^{+}75$ - $^{-}20$ =

(c) $^{-}43$ - $^{+}68$ = (d) $^{-}99$ - $^{-}153$ =

(e) $^{-}765$ - $^{-}219$ = (f) $^{-}810$ - $^{+}810$ =

(g) $^{-}230$ - $^{+}105$ =

Results of Instruction: Due to lack of time, only Connie, using rules, completed these tasks, getting most of them right, except for subtractions like (a) and (d) where she kept the sign and subtracted. Jason and Steven were able to solve the examples which they attempted, but Robert needed help on all but one, and when the lesson ended, he still had not grasped the subtractions that needed an equivalent representation of the initial number. In retrospect, more time should have been spent with him using the concrete material.

It appears that Jason's number line strategies were now possibly being connected with chip neutralization, as follows in this example:

Jason: ($^{+}100$ - $^{+}145$) "There's a hundred. Just forget the 45. There's a hundred, a hundred take off a hundred, is zero, there's 45 extra, so it's gonna be negative 45."

INDIVIDUAL STUDENT PROGRESS

Jason was the only student who was unable without intervention to remember and use the notions of neutralization within the concrete model. He rarely chose to refer to procedures from the model, although once reminded, he could perform them when necessary. He seemed to have abstracted notions learned from neutralization, and applied them to number line references in order to succeed on more types of subtraction tasks.

Connie was the most successful on all subtraction tasks, applying rules that she constructed from observing number patterns within the lesson rather than from the physical model, but lacking in understanding of the meaning of integer subtractions. Both she and Jason still trusted results obtained by chip manipulations over their own mental reasoning.

Steven found the expanded subtraction sentences easiest to deal with, but never used the concepts to solve mentally - he seemed to need the visual evidence that the procedure was correct.

Robert had the most difficulty with the subtractions. He did not seem to find either the chips or the expanded sentence an intuitive way to solve the given tasks.

EVALUATION OF LESSON CONTENT

The neutralization model provided meaning for "trivial" subtractions such as $^{-}9$ - $^{-}5$ (Robert: "Say you have nine of these (chips), then after you're taking five of them away") and -n - -n

(Steven: "You're minusing the same thing"), as all students were able to solve them intuitively. However, non-trivial integer subtraction did not appear to be demystified by the neutralization model. In order to understand these subtractions where there were not enough to take away, the students had to:

a) recognize that there was a problem with respect to quantity, which all students did.

b) believe that a solution of some type <u>does</u> exist (this is a feature of this new set of numbers). This was not expected to be known, since they did not yet know how the integers behave under subtraction, but did know that not all subtractions are possible in N.

c) recognize that an equivalent representation of the initial integer would solve the dilemma, which is a demanding task, and was not constructed by anyone from previous knowledge about equivalence classes of integers, but was accepted by all of the students when guided in that direction.

d) accept that the equivalent representation was an entire group. This was spontaneous for both Robert and Steven, but both Jason and Connie believed that it was two groups, zero and the original number, but were led to accept it.

e) know which representation was the appropriate one for the particular problem, which was another demanding task. This continued to be a difficulty for all four children throughout the instruction, especially when moving from one type to another (both signs same, both signs different).

f) be sure to put in both the positive and the negative parts of the appropriate neutral group, and not be distracted by the nature of the subtrahend (procedural understanding). Each student occasionally neglected to do so.

g) know that "taking away" is performing the required subtraction, which all students understood intuitively

h) make sense of the results of the subtraction. Only Jason, who used integers as positions on the number line could make sense of all results. Connie had no context for her results, but only sought patterns she could use to obtain an answer, even though integers had become identified with quantities of chips. Steven and Robert concentrated on manipulating the numbers within the expanded sentences rather than on the results of the exercise.

In defense of the model, the author was too sure that both kinds of "not able to subtract" tasks would be considered similar by the student, but in retrospect, it seems that the cases should have been separated more clearly, and the students should have been given more practise with each, going to larger number examples to ensure that the ideas were assimilated. The instruction should have began with subtractions with unlike signs, where the number of neutrals to be added is the same quantity as the subtrahend, to eliminate the need to calculate how many to add. As it was, this notion of subtraction became more of a procedural manipulation of changing to an appropriate equivalent representation and seemed to focus more on the _additive_ nature of including neutrals than on the removal of the desired amount of chips.

It appears that within the neutralization model, subtraction _is_ harder than addition, and more than one lesson should have been used to master these procedures, with much more discussion when the student was still at the physical level. At the end of the addition lesson, the students were able to verbalize rules for different types of additions, but at the end of this lesson on subtraction, only rules for "trivial" subtractions had been formulated. Jason is the only one who was able to make some generalizations (based on understanding of number line positions, perhaps influenced by neutralization ideas).

7.2.4 LESSON FOUR: Ordering

This lesson was a shorter one, and focused on the review and consolidation of subtraction from the previous lesson, and on ordering of integers by position as well as the "absolute value" ordering that had been taking place with integer tasks.

(1) SUBTRACTION REVIEW

The students were given the following subtraction tasks with 2 and 3 digit numbers, and were asked to explain their solving strategies. If any student could not subtract correctly, he was first given a similar one-digit task, and if this was too difficult, then the chips were used to solve it. Extra questions were given if anyone needed them.

(a) ${}^{-}18 - {}^{-}58 =$	(b) ${}^{-}80 - {}^{+}75 =$
(c) ${}^{+}30 - {}^{-}30 =$	(d) ${}^{+}89 - {}^{-}20 =$
(e) $0 - {}^{-}45 =$	(f) ${}^{+}30 - {}^{-}95 =$
(g) ${}^{-}81 - {}^{-}81 =$	(h) ${}^{-}13 - {}^{+}49 =$
(i) ${}^{+}42 - {}^{+}100 =$	(j) ${}^{-}27 - {}^{+}27 =$

Results:

Steven had no difficulty with any of the subtractions. He chose to use the expanded mathematical sentence as his solution procedure because he termed it "easier" to use. He had always written the addition of neutrals (when the signs were different) in such a way that the second neutral matched the subtraction, so that he always cancelled the last two numbers and added the first two. For example, for + 89 - -20, he wrote ${}^{+}89 + {}^{+}20$ **$+ {}^{-}20 - {}^{-}20$**. On the fourth example, he declared that he did not need to complete the sentence, and that "as soon as I have the first two numbers, I know it (the answer)", but he did not at any time notice how this shortened version related to the original subtraction task. He solved the next two problems mentally, using neutralization ideas (this was the first and only time he had solved mentally except for trivial subtractions). Only task (i) ($^{+}42 - {}^{+}100$) caused him to revert back to writing the whole sentence, since the cancelling was of a different type.

Jason also was able to solve all subtractions correctly. The first task ($^{-}18 - {}^{-}58$) was one which made sense for him on the number line, but the second one ($^{-}80 - {}^{+}75$), which was of a type that he had previously been able to justify as going farther below zero, caused him to say "I <u>think</u> I have to add cause I <u>can't</u> subtract." At this point the instructor referred verbally to the parallel chip situation to justify the addition, and from then on, he used the notion of the chips as a mental model to solve all other tasks of the type where the signs were different.

Robert had forgotten how to do any subtractions, which was an anticipated reaction, given his confusion during the previous lesson. He needed to be given an example with the chips, and needed help to construct the expanded sentence to show the subtraction. This help was needed for the first three tasks, after which he was able to correctly use the sentence for all other examples, even to the extent of perceiving that for $^{+}42 - {}^{+}100$ the addition of 100 neutrals was not the best way to carry out the subtraction, changing the sentence without any intervention from myself.

Since Connie had been devising rules based on patterns during the previous lesson, she was first given the following series of tasks using only small numbers to encourage her to make observations about the physical model which she could apply to integers so that she would possibly have a mental model as the backup for her rules.

(a) $^{+}4 - {}^{+}10$	(b) $^{-}2 - {}^{+}3$	(c) $^{-}4 - {}^{+}7$	(d) $^{+}9 - {}^{+}10$
(e) $^{-}6 - {}^{+}4$	(f) $^{-}2 - {}^{-}11$	(g) $^{+}9 - {}^{-}5$	(h) $^{-}1 - {}^{+}1$
(i) $^{-}5 - {}^{-}3$	(j) $^{+}7 - {}^{+}2$	(k) $^{+}11 - {}^{+}20$	

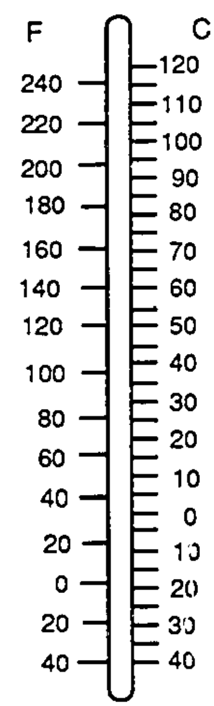
When specifically asked to use the chips to solve with, she was able to do so correctly, although always adding more neutrals than necessary, but as soon as she was told to use any method, she reverted to looking at the signs of the numbers to determine her procedure. She verbalized that the sign of the leading integer gave the sign of the answer and of what operation to perform, although sometimes the operation was determined by whether or not the signs were the same (if not, subtract). These "rules" were applied very inconsistently. Whenever there was an incorrect answer (on half of these tasks), she was asked to verify the answer with the chips, and although she trusted the chips answer over her original answer, she never used the chips on her own initiative to find the answer to any subtractions. Twice she referred to the chips mentally, although since she described adding in too many neutrals, it is not clear how she dealt with their presence. For example, for $^{+}9 - {}^{+}10$, she said, "you have 9 positives and you put 10 neutrals in, with the negatives, then that will leave you with a negative 1, cause there's none left".

On the tasks with larger numbers, Connie again had 50% of them incorrect, due to her incorrect reasoning. Whenever the magnitude of the numbers was the same, she tested to see if neutralization was occurring (for example she correctly solved $0 - {}^{-}45$, (because "positive 45 plus negative 45 equals zero".) and $^{-}27 - {}^{+}27$ because it was not the addition of opposites, yet she rejected her original answer of zero for $^{-}81 - {}^{-}81$ because it did not fit the pattern of addition of opposites. When chip subtraction was returned to at the end of the tasks, to see if the instructor could help her connect the chips to the integer tasks, she asked for the "trick": "Is it, is the, you know, the trick, is it when you see an addition sign, you add them?" (it is interesting that she has been interpreting a positive sign to mean addition). It is here that she was told that the trick is to check to see if subtraction is possible, and if not, one must add neutrals in order to perform the subtraction. This is the exact procedure that she had been performing correctly with chips, but had not connected to numerical situations. She was given three more examples with large numbers ($^{-}100 - {}^{+}30$, $^{+}312 - {}^{+}500$, $^{+}64 - {}^{-}20$) which she then correctly solved using this procedure as a mental model.

(2) ORDERING OF INTEGERS BY POSITION ON AN EXTENDED NUMBER LINE

It was reported earlier that the boys had all been able to draw and correctly label an extended number line in the interview, but had each used three separate rules (for pos-pos, neg-neg, and pos-neg) to order by position within the various parts. Connie was inconsistent in her drawing of a number line, and ordered all integers as natural numbers only. Thus the objectives in this part of the lesson were somewhat different for the boys than for Connie. All four students needed to understand that the integers formed a set of numbers, and thus one ordering rule would suffice. Here the term "integer" was introduced for the first time.

In order to bring their real-life knowledge into play they were shown this picture of a thermometer with a double scale (to reinforce the arbitrary nature of the placement of zero) and asked how to determine hotter or colder temperatures. It was decided not to use an altitude example, since it may have reinforced the historical obstacle of the divided number line, split at zero. This discussion led into the left-right determination of greater than or less than criteria for the "mathematical" number line.



Connie however, was first asked to draw a number line to determine her thinking about the placement of the numbers, and to help her to construct the correct positioning of integers on the line. Connie was also asked to play the game "war" with integer cards from $^{-}15$ to $^{+}15$, where she and the instructor would each turn over the top card of a pile, and the person with the greatest integer on their card would keep both cards, unless both had the same value. Connie was told that she would make all the determinations herself, and that if she were wrong, she would have to give up two extra cards as a penalty.

Results of Instruction:

Steven and Jason were able to consolidate their own rules for ordering by position on sections of a number line into one which takes the one on the right at any place on the line as the highest. Robert agreed when told, and Connie was the only one who had no previous ordering rules with which to substantiate this, but was able to construct it with intervention. She was unable to draw a number line correctly until put into a conflict situation (no room for extra negative numbers). After instruction, she was always able to correctly determine the "greater" numbers in the "war" game. In retrospect, it would have also been good to play the game again, using the "lesser number" as criteria for winning each round.

(3) ORDERING BY MAGNITUDE

The students were asked if they thought that a particular negative number (for example $^{-}10$) could ever be considered to be larger than another particular number (for example $^{-}2$). Discussion of quantity of chips, or decision making for additions and subtractions was used to consolidate this view, and the term "absolute value" was introduced.

Results of Instruction:

This type of comparison of integers gave no one any difficulty.

INDIVIDUAL STUDENT PROGRESS

Connie easily learned ordering by position, which should strengthen the correct positions of the negative numbers for her. It may also help her to put the negative numbers into another context (able to be ordered) in order to help her accept them as "existing" as numbers. It is not clear whether or not the students will have overcome the obstacle of the "divided number line".

7.2.5 LESSON FIVE Abstractions

Since this lesson would focus on the relationships between addition and subtraction of integers, the students were first presented with a mixture of these tasks for the first time.

(1) REVIEW OF PREVIOUS PROCEDURES

(a) Mixture of Addition and Subtraction tasks: The students were asked to solve the following tasks. If they had forgotten strategies, or were confused between addition and subtraction, they were given parallel situations with one digit numbers, and were allowed to used chips to solve if necessary.

(a) ${}^{+}66 + {}^{-}21 =$ (b) ${}^{-}85 - {}^{+}34 =$

(c) ${}^{-}95 - {}^{-}18 =$ (d) ${}^{+}78 - {}^{+}120 =$

(e) ${}^{-}99 + {}^{-}53 =$

Results: This was the first time they had been confronted with a <u>mixture</u> of operations, and for Connie and Steven, the last instruction of lesson four had occurred several days previously (Steven 5 days, and Connie 4 days). The first task was an addition one, and all but Steven (who "neutralized") subtracted (i.e. mentally added in neutrals so there would be enough to remove). Jason was able to recover, and "neutralized" when the operation was stressed, but both Robert and Connie had to be given a simpler example, and eventually the chip situation. All students regarded (c) and (e) as trivial operations, and had no difficulty.

Jason correctly solved all other tasks, using the chips mentally for (b) and a mental number line in a neutralizing fashion for (d).

Steven solved (b) and (d) by writing the extended number sentence, but added an incorrect number of neutrals twice in (d) and needed to be confronted with a one-digit situation using chips in order to solve correctly.

Robert had no difficulties with (b) and (d) once he was reminded that he should look at the operation sign.

Connie initially tried to remember a rule for solving (a) and (b) and couldn't, but could solve correctly using chips, although she was not always able to make the connection between the chip situation and the larger number task. In (d) her problem with adding a random number of neutrals surfaced, as she said she could add "about 50". When she was asked if there was a way to know for sure how many to add, she just subtracted, and solved the problem mentally.

(b) Missing Addend in Decompositions: To check procedures for finding the missing number in additions or subtractions, the students were asked to solve the following tasks:

(a) $345 + \square = 800$

(b) $582 - \square = 106$

Results: The set of tasks were solved with no difficulty by all students, all but Steven using rearranging and subtraction to solve. Steven subtracted in the second and added on in first, using subtraction to check his answer.

(2) REVERSIBILITY OF ADDITION AND SUBTRACTION OF INTEGERS

In order to evaluate whether or not the reversibility of addition and subtraction would be seen to hold for integers, the students were given the following tasks to solve, divided as follows:

(a) the first two are "trivial" and can be solved by subtraction with notions from N,

(b) the next three could be approached from a "missing addend" point of view within Z (positive 6 plus what gives positive 4?),

(c) the last requires a sense of outcome of operations in Z.

Any of the questions can also be solved by rearranging so that the given numbers may be operated on with known procedures (for example, the last task could become $^{-}8 - {}^{+}7 = \square$).

(a) $^{+}45 = {}^{+}16 + \square$ (b) $^{-}82 = \square + {}^{-}20$ (c) $^{+}6 + \square = {}^{+}4$
(d) $^{-}12 - \square = {}^{-}3$ (e) $^{+}5 + \square = {}^{-}2$ (f) $^{-}8 - \square = {}^{+}7$

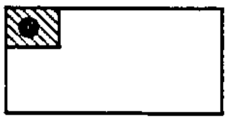
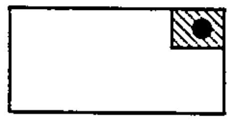
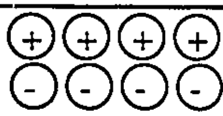
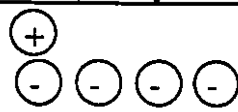
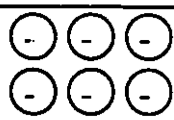
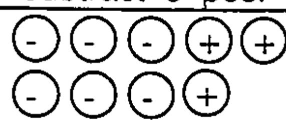
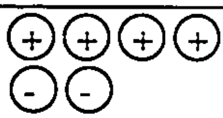
Results of Instruction:

The first two tasks were solved by reversibility, except by Steven, who added on. The next three tasks were all solved by adding-on techniques by all students. Robert and Connie did not immediately solve (e) correctly, Robert searching for something to neutralize, and Connie trying to figure out what rule to use. The last example proved to be difficult for all but Jason, who used the number line concept of having to go above zero by a certain distance. The others tried to use substitution, and for Steven in particular, the awareness of results of operations was greatly lacking (yet for him, this skill was strong for whole numbers). All of the students were eventually able to solve the problem, with chips to validate their attempts and confirm their answers.

In retrospect, this was not well tested, and tasks could have been developed both with large and small numbers, with more combinations of signs, and with the missing number in different positions (using the forms of $\square + n = m$, $n + \square = m$, $\square - n = m$ and $n - \square = m$). This would have helped in particular to develop an awareness of results of operations. It was seen that the reversibility of addition and subtraction was not spontaneously used by these students in these particular integer problems, although it had been intuitive for them to use it for the whole number tasks.

(3) MORE THAN ONE WAY TO NEUTRALIZE

In order to evaluate whether or not students realized that neutralization could be accomplished in more than one way, and to develop this thinking, the following tasks were presented, and after one solution, the student was asked, "Is there any other way this could be neutralized?". Although tasks similar to this were not given with the physical chips (due to the neglect of the instructor), the examples with the charged particles served as a replacement.

Initial State	Change	Neutralization
	 add 60 kg to left	____________
	 flip right	____________
	 subtract 3 pos.	____________
	 add 3 positives	____________
	 subtract 2 negs.	____________

Results of Instruction:

Steven spontaneously volunteered two ways to neutralize (restore to the initial state) for three out of the five tasks (the first, third and fifth), and supplied a second way when asked for the other two. Robert spontaneously gave two ways on the last task, after giving a second way when asked for all others except the flipped shape. The other students always needed to be asked for a second way. Connie's answers were sometimes more than the minimum necessary, (for example, for the shape which had to be flipped, she said "you flip it down, and then to the side, and then you flip it up"), indicating a freedom from "formula" responses. Jason needed assistance with two of the tasks - the flipped shape, and the last task. It seems unlikely that Connie or Jason would have formed any notion of a relationship between the two ways used to neutralize: Connie, since her second strategy was sometimes not the opposite of the first, and Jason, since he seemed to "solve again" the second time. Neither Steven nor Robert verbalized any relationship between their two solutions.

(4) EQUIVALENCE OF THE OPERATIONS OF ADDITION AND SUBTRACTION

The students were presented with a number of mathematical tasks, covering several mixtures of signs and relative size of integer, of returning to an initial state by addition and/or subtraction, as shown below. They were asked to write the number sentence that showed the return to the original integer.

Start With	Operation	Neutralization
310	add 99 310 + 99 = 409	
$^{+}12$	add $^{+}5$ $^{+}12 + {}^{+}5 = {}^{+}17$	
$^{-}4$	add $^{-}10$ $^{-}4 + {}^{-}10 = {}^{-}14$	
$^{+}11$	subtract $^{-}3$ $^{+}11 - {}^{-}3 = {}^{+}14$	

$^{-}9$	subtract $^{-}1$ $^{-}9 - {}^{-}1 = {}^{-}8$	
600	subtract 150 $600 - 150 = 450$	
$^{+}7$	subtract $^{+}10$ $^{+}7 - {}^{+}10 = {}^{-}3$	
$^{-}1$	subtract $^{+}5$ $^{-}1 - {}^{+}5 = {}^{-}6$	
$^{-}6$	add $^{+}1$ $^{-}6 + {}^{+}1 = {}^{-}5$	
$^{+}9$	add $^{-}7$ $^{+}9 + {}^{-}7 = {}^{+}2$	

They were always asked if they saw a second way to achieve the result, and were allowed to do as many examples as was necessary until they either commented on the nature of how the two ways related to each other, or until they seemed to construct the second way rapidly, upon which they would be asked how they formed the solution so quickly. When they had made some sort of "rule", it was tested on further examples, of both the instructor's and the child's making, until the student was convinced that the rule would always hold.

The power of that relationship was then exploited, as the student was confronted with the two ways of performing a task, using the following examples, and was asked to evaluate which of each pair of operations would be "easier" to solve, and give their reasoning.

$^{-}10 + {}^{+}2$	$^{-}6 + {}^{-}2$	$^{+}3 + {}^{+}1$
$^{-}10 - {}^{-}2$	$^{-}6 - {}^{+}2$	$^{+}3 - {}^{-}1$

They were then asked to solve some two digit problems, first rewriting the problem in a second way, then evaluating which way was easier, and checking their answer by evaluating the second way. The following tasks (or similar ones) were used:

(a) $^{+}80 - {}^{-}65$ (b) $^{-}29 + {}^{-}48$
(c) $^{+}195 - {}^{+}218$ (d) $^{+}82 + {}^{-}58$

Results of Instruction:

All students almost exclusively sought first to reach the original number by "undoing" the operation which had been performed on that number. Robert entered into this activity by spontaneously finding two ways to reach a common target, perhaps due to the activity above. When asked how he was able to find the second solution so quickly, he responded that either taking a number away or adding on the opposite would have the same effect (again, a conclusion that may have come from the previous activity). He thought that this would always be true, and was able to come up with his own example to test his thinking. His focus seemed to be on the operations involved.

Although Steven had come up with two solutions on his own to the previous exercises, in this task he was only able to quickly "undo" the given operation. When pushed to find another way, his strategy was to try to find another way to use the same "undoing" operation with a different representation of the number to be operated by (for example, the subtraction of $^{+}10 + {}^{-}5$ (a second strategy to neutralize) was described as the same as the subtraction of $^{+}5$ (the first strategy)), and he had to be specifically asked to add something. His focus was on the target desired, and he sometimes suggested a second strategy that would not bring him near that target, showing a lack of awareness of the results of operations. On the fourth example ($^{+}11$ - $^{-}3$ = $^{+}14$), where his first solution was to subtract $^{+}3$ from $^{+}14$ to get $^{+}11$, he tried using the opposite "signs", as he "wanted to see" if this would produce the same answer (he said that he made that rule up at home), yet he admitted that he did not know if the results would be the same. Here, his vocabulary indicated that he was focusing only on the signs, without regarding the switch as an operation change and a change to the opposite integer; in other words, his understanding was procedural only. Once his "rule" was confirmed with one example, he trusted it completely.

Jason could think of "no other way" without intervention for four of the first five examples, and in fact needed assistance in finding the first solution to the fourth and seventh task. He usually focused on the operation rather than on the target to get the first solution. After the seventh example, he stated that you could exchange two "signs" that were the same (plus a positive, subtract a negative), and needed intervention to notice that this exchange had also been true for opposite "signs" as well.

It was expected that Connie might not be ready for this activity, since she still had not been able to consistently solve additions and subtractions, but she showed an ability to understand the effect of operations (although interventions were necessary for the first integer example). She began by "undoing" the given operation as her first answer, and reasoned about how to obtain the desired result for her second answer. She herself noticed that the sign of the number and the sign of the operation had changed (after three examples), but did not necessarily connect this information with subsequent tasks.

According to many authors who are concerned with integer operations, it should be easy for students to see that subtraction is equivalent to the addition of the opposite number, yet this was not the case with Jason, who seemed to see the finding of a second way to solve as an unrelated task, and Connie, who would perhaps use the rule (if she were able to keep track of when it applied), but not need to understand why it worked. It is interesting that Steven had apparently discovered this rule some time ago, yet did not spontaneously see that this exercise was an application for the rule. Robert seems to be the one with the most understanding of the reasoning behind the rule, and would likely use it with understanding, being able to substantiate it.

In retrospect, Jason, Connie and Steven may not have been given enough examples to consolidate their construction of this notion and to use it in an appropriate setting. This also seems to be a notion which should develop more intuitively over a long enough period of exposure to additions and subtractions to ensure an awareness of the results of these operations.

The three boys were all able to evaluate that the ability to change the sign of the operation and the sign of the subsequent integer would usually result in one of the operations being much easier to solve than the other. They were all able to make the change, and to solve both ways to confirm that the answers were the same.

(4) RESULTS OF OPERATIONS

The students were presented with the following "opinion poll" and asked to choose the best completion to the sentence, then to back up their choice with examples. If the wrong choice was made, after the example was given, the student was still asked if he were sure that there was no other possibility, and if the student was still convinced, a counter-example was given.

When I add two numbers, my answer is
___ (a) always bigger than both numbers
___ (b) sometimes bigger than both numbers
___ (c) never bigger than both numbers (always smaller than both numbers)

When I subtract two numbers, my answer is
___ (a) always smaller than the first number
___ (b) sometimes smaller than the first number
___ (c) never smaller than the first number
(always bigger than the first number)

Results of Instruction:

Robert was the only student who had already decided that addition and subtraction could be used to make either smaller or larger, and was able to give examples of all but subtraction making larger. His examples were: 40 + 48 = 88; 48 + $^{-}$48 = 0, 8 - 6 = 2 and $^{-}$8 - $^{+}$6 = $^{-}$14.

Steven initially stated that addition would always make bigger, but "because of negatives" decided that it would be 'sometimes'. He felt that subtraction should also be 'sometimes', but he could not come up with an example of a larger result. His examples were 2 + 4 = 6; $^{+}$2 + $^{-}$5 = $^{-}$3; 2 - 2 = 0; 2 - $^{+}$5 = $^{-}$3 and $^{-}$2 - $^{+}$5 = $^{-}$3.

Jason continually refined his opinion as he went along, first saying that addition would 'sometimes' make larger, then after giving the examples of 6 + 6 = 12 and 6 - 6 = 0 and being reminded that the operation was addition, stated that addition would always make larger, but then decided that he could add a negative, which changed his opinion again. He said that subtraction would always make smaller, yet his example ($^{-}$4 - $^{-}$3) proved him wrong, so he switched to "probably sometimes", but could not give an example of subtraction making smaller.

Connie was totally confused (and tired). She could no longer even determine bigger and smaller, and could not perform the operations (e.g. $^{-}$5 + $^{-}$7; $^{+}$8 - $^{-}$5).

INDIVIDUAL STUDENT PROGRESS

Connie had difficulty with procedures and outcomes of any non-trivial integer additions and subtractions. Steven and Robert were able to appreciate the reversibility of operations, yet Steven had very little awareness of results of operations. Jason needed more time to consolidate these notions.

EVALUATION OF LESSON CONTENT

This lesson seems to have been premature, since the students needed to spend more time in the integer environment before they could construct abstractions about these numbers and about results of operations on integers.

Reversibility of addition and subtraction had developed through the idea of neutralizing an action that had been performed where one operation was seen as the "undoing" of the other.

Presentation of the "subtraction rule" (add the opposite) in the context of it being an "optional" strategy that could change a question into an easier one seems a more natural way to approach this idea than the traditional one, since it is then not a "rule without reason", but a rule with a purpose. This introduction to the rule is of course not specific to the neutralization model.

It seems that the results of operations were not adequately observed, as especially a bright student like Steven had not developed this understanding from simply performing several operations. Perhaps this could have been constructed more intuitively from tasks such as the following, which do not require solving, but thinking about what range the answer should be in:

What is the <u>approximate</u> answer to $^{+}382 + {}^{-}70 = \square$

(a) $^{+}450$ (b) $^{+}300$ (c) $^{-}450$ (d) $^{-}300$

and What is the <u>approximate</u> missing number? $\square + {}^{-}92 = {}^{+}34$

(a) $^{+}120$ (b) $^{+}60$ (c) $^{-}60$ (d) $^{-}120$

7.2.6 LESSON SIX Post-test

The students were given a post-test using a semi-standardized interview format, approximately two to four days after lesson five. There was no opportunity to do a follow-up post-test at a later date.

(1) ORDERING OF INTEGERS

The students were asked the following general question to see if they had accepted the notion of integers as a complete set with one rule governing number line ordering, and if they realized that ordering by absolute value was also a valid procedure to perform on integers.

If I gave you any two integers, and I asked you which one was larger, how would you decide?

If the student gave an incomplete rule (restricted to one section - either the positive or negative section - of the number line) they were challenged by counter examples to see if they would give an overall rule, or three separate ones.

Following this discussion, they were asked to put the following integers in linear order:

$$^{+}7,\ ^{-}8,\ 0,\ ^{-}20,\ ^{-}2,\ ^{+}3$$

To evaluate their notion about the neutrality of zero, they were asked the following:

Why don't we write a positive or negative sign on the zero?

Results:

(a) Deciding how to order two integers:

Connie was the only student who spontaneously answered that she'd see "which one's more right". This is probably a result of having no previous notion of number line ordering before the instruction. All three boys initially gave conditions on their ordering rules, restricted to specific sections of the line, and after intervention with counter-examples, both Jason and Robert were able to state that one could use the number on the right for any two integers, but Steven said he couldn't think of any rule that would work for all integer situations. Robert was the only student who mentioned that one could also order by quantity ("how much you have of each integer").

(b) Ordering Integers

No one had any difficulty in ordering the given integers linearly.

(c) Zero as Neutral

Robert and Steven explained that zero has no sign because "it's been neutralized", and "it's neither positive nor negative, it's in the middle, plain zero". In lesson two, when opposites were being added, Steven jokingly gave an answer as "a negative positive zero", and when asked if zero gets a positive or negative sign, he replied, "neither, it's neutral".

Jason said that it could have either sign put in front of it, "it doesn't matter", and Connie said that it was "a negative and a positive, it is in the middle of the number line, it could be either on the negative side or on the positive side". These two students also had this idea of neutral in the first lesson, where they said that the neutralized drink would be both sweet and sour. It is possible that the physical representation of the neutral element, with both its positive and negative components, could foster this idea that zero is both negative and positive.

(2) INTEGERS AS OPPOSITES

The students were asked to give the opposite of $^{-}216$, and to explain why it was opposite.

Results: All four students correctly identified +216 as the opposite, Jason and Steven using the concept of above and below zero, Robert referring to having something and missing something, and Connie using the notion that positive and negative are opposites. None of the students mentioned the idea that the two integers combine to make zero, or to neutralize.

(3) RULES FOR OPERATIONS ON INTEGERS

Before given any specific addition or subtraction tasks, the students were asked for general strategies for these operations to see whether or not they had formulated any rules for these operations. The following questions were asked of each, and if only one type of situation was covered by the rule, they were again reminded that they must think of <u>any</u> two integers - would their rule work for a different situation?

> If I gave you any two integers, and I asked you to <u>add</u> them, what would you do? What would your thinking be? (What would you look for? What would be your strategy? etc.)
>
> If I gave you any two integers, and I asked you to <u>subtract</u> them, what would you do? What would your thinking be? (What would you look for? What would be your strategy? etc.)

Results:

All of the students were able to verbalize the following two addition rules: (1) if the signs are the same, keep the sign and add the quantities (2) if the signs are different, neutralize first, then use the remainder. They were also able to give three subtraction rules: (1) if the signs are the same, and the first number is "larger", just subtract (2) if the signs are the same, and the first number is "smaller", add neutrals, then subtract (Jason would "go to the other side of zero" in this case), and (3) if the signs were different, add neutrals, then subtract. Steven also gave the rule (through an example) that addition of opposites cancel out and equal zero.

It was surprising that Connie was able to (a) verbalize these rules, since she often had difficulty in giving explanations (b) know these rules, since she did not often seem to be using them in concrete examples for which she would try to figure out some kind of sign manipulation (c) give these rules with no reference to any examples (all three boys needed numbers at some point), yet perhaps particular examples distract her from the general notions.

All three boys pushed to use concrete integer situations, although Robert held off at the instructor's request as long as he could (until subtraction when neutrals had to be added). Jason made up his own examples for each case to describe what he would do to solve them, and Steven kept asking the instructor to "pop two things in your head", so that he could solve them, but when his request was declined, he supplied his own.

(4) CANCELLATION OF INTEGERS AND WHOLE NUMBERS IN A STRING

The following two tasks were given to the students, with no instruction to cancel, only to solve. The integer example contains an example of the addition of opposites ($^{-}45$ and $^{+}45$), the subtraction of the same sign (-20) and the non-cancelling subtraction of the opposite ($^{-}12$ - $^{+}12$). The whole number task was given just to evaluate whether or not the instruction on cancellaton was beneficial in helping the students to overcome their difficulties with this type of task.

(a) $^{+}61 + {}^{-}20 + {}^{-}45 + {}^{-}12 - {}^{-}20 + {}^{+}45 - {}^{+}12 =$ ______

(b) $926 - 78 + 300 + 54 - 82 - 54 - 82 - 926 + 78 =$ _____

Results:

(a) Integer Cancellation

None of the four students cancelled correctly, as they were considering only the sign of the numbers (Jason and Connie) or the sign of the operation (Steven and Robert) even though they had earlier been instructed to take both into consideration. Jason and Connie cancelled the 45's and the 12's since they were opposite numbers, and would neutralize. They rejected the cancellation of the 20's since they were both negative, and both obtained +61 as their answer (Connie using absolutely correct procedures to first add on ⁻20, then subtract it!). Robert cancelled the 12's and the 20's because they were added on, then taken away, and he also noticed the "neutral" 45's, again obtaining +61 as the answer. Steven was the only student who obtained the correct answer, but he did not cancel the 45's. Although he had only been focusing on the sign of the operations, he happened to notice that the 12's did not cancel because subtraction of the same number was not taking place. He rearranged in order to calculate (first he tried to group all the positives and ran into trouble), then did 61 - 12; + 45; + ⁻12; + ⁻45.

(b) Whole Number Cancellation

Both Robert and Steven correctly cancelled and solved this exercise, as their strategy of focusing on the operation gave them success here. Neither one had any problem with detachment of any sign. Connie just cancelled the numbers that were the same, with no regard to the operations, and obtained 300 as her answer, and Jason cancelled the minus sign following the leading 926 as he cancelled that number, so he felt that the 78's would not cancel since they were now both positive. He was left with the 300, the 78's and the 82's, and was ready to calculate the result, then reconsidered, and cancelled the 82's.

(5) INTEGER OPERATIONS

The students were asked to solve the following integer tasks, while explaining their thinking, with no intervention given unless there seemed to be carelessness involved (omitting a sign for example). These tasks represent 8 of the subtraction categories and 5 of the addition categories.

(a) $0 - {}^{-}300 =$	(b) ${}^{+}219 + {}^{-}19 =$	(c) ${}^{-}906 - {}^{+}906 =$
(d) ${}^{+}78 - {}^{+}100 =$	(e) ${}^{-}218 + {}^{-}51 =$	(f) ${}^{-}280 - {}^{-}280 =$
(g) ${}^{-}86 - {}^{+}21 =$	(h) ${}^{-}817 + {}^{+}817 =$	(i) ${}^{-}390 - {}^{-}105 =$
(j) ${}^{+}62 - {}^{-}50 =$	(k) $0 + {}^{-}95 =$	(l) ${}^{-}820 + {}^{+}1000 =$
(m) ${}^{+}342 - {}^{-}342 =$		

Results:

Robert solved all tasks correctly, writing the expanded integer sentence for subtractions requiring the addition of neutrals, except for examples (k) and (l) where he mentally used the same notions. Jason made one arithmetic error, and solved by neutralizing the addition of opposite signs, and by "thinking of the chips" to solve subtractions where neutrals could be added, except for task (d) where he was able to "go below zero".

Steven made a lot of careless mistakes: not checking to see whether the question was an addition or subtraction task, leaving off the negative (or positive) sign, arithmetic error. He wrote no expanded subtraction sentences, but used the chip ideas mentally. Tasks (b) and (l) were both addition of unlike signs, and while he neutralized the latter, he used his rule of changing the signs on the former, where the signs became the same and the operation was a trivial subtraction.

Connie was more successful than ever before, yet still made errors. She tried to use the chip ideas mentally, and had problems when the subtractions were the same sign but needed neutrals to be added in. She also felt that (c) would be zero because of neutralization, neglecting to observe the sign of the operation.

(6) PROCEDURES OF ADDITION AND SUBTRACTION OF CHIPS

The students were asked to show how to add ${}^{-}8 + {}^{+}3$ and how to subtract ${}^{+}6 - {}^{-}4$ using either chips or written plus and minus signs.

Results: Steven was the only student who experienced difficulty here, trying to add neutrals in the addition problem.

(7) EQUIVALENCE CLASS OF INTEGER NOTATION

The students were asked to draw three different ways to represent the integer $^{-}2$.

Results: Only Jason was not able to make any other representation than two negative signs.

(8) EQUIVALENCE OF OPERATIONS

The students were given the following two situations, one at a time:

> I started with positive 3, and I ended up with negative 2. What did I do?
>
> I started with negative 9, and I ended up with negative 5. What did I do?

When they had given one solution, they were asked if there was any other way that I could have obtained the given result.

Results: Steven was the only student who saw the correct two ways for both problems. For the first one, he added $^{-}5$, then said that he could also subtract $^{+}5$, because of his experience with a card game (Golf) where the Joker means that 5 points are subtracted from your score, and sometimes your score becomes negative (thus his second solution was unique to the situation of the example). For the second task, he initially subtracted $^{-}4$ (the first strategy of all 4 students), then said that he could also add $^{+}4$, because of the sign change rule done in another session. Connie also used the rule for the second example, and Robert obtained the second answer with no reason given (he had not been able to give a second way in the first task). Jason, however, could see no second solution for either, even when he was told that $^{-}4$ had <u>not</u> been subtracted in the second case, and that there had to be another way. Connie had been unable to give a correct solution to the first task (she wrote $^{+}3 - {}^{-}1 = {}^{-}2$), and could not see another way, but when she remembered the rule during the second task, she went back and gave $^+3 + ^+1$ as her second solution, then saw that it gave her $^+4$ rather than the desired $^-2$. This led to attempts with systematic substitution, and she finally came up with $^+3 + ^-5 = ^-2$, then said you should be able to subtract to get a second solution (using the rule), but didn't know what to subtract. This leads one to hypothesize that the rule was easier for them to access when the sign of the operation and the sign of the number were the same (as in the second task) where two positives are exchanged for two negatives for example, but in the first task, where it was one of each, perhaps the change is not so obvious.

INDIVIDUAL STUDENT PROGRESS

Robert was able to solve all tasks except for the cancellation of integers, where he focused only on the sign of the operations. He still used the expanded sentences for the integer subtractions. While he knew the equivalence of operations, he never used this technique, but this was consistent with his usual approach, since he rarely saw an easier way of performing a task until after he had solved it from left to right.

Steven was careless in some of the tasks, yet when brought to observe more closely, was able to see his errors. He still used separate rules for ordering on the number line, and had difficulty with the integer cancellation. He used the equivalence of operations twice, knowing that he could trust it.

Jason finally used the neutralization model as a mental model to solve the types of tasks that he was unable to solve before our sessions together. Even though he added neutrals to solve the subtractions, he gave no equivalent representations for an isolated integer, which suggests that he may still have regarded the addition of neutrals as a set separate from the given integer. He still had problems with cancellations of both whole numbers and integers.

Connie finally began to use the ideas of the neutralization model to solve operational tasks, but there is no indication that she would continue to do this on another occasion, since her learning always seemed to be reconstructed in each session, with new ideas replacing old ones, not enhancing them. She was only able to work with one factor at a time, so the cancellation tasks were beyond her ability.

CHAPTER EIGHT

CONCLUSIONS

AND RECOMMENDATIONS

8.0 EVALUATION OF TEACHING EXPERIMENT

The teaching experiment will be evaluated with several concerns in mind. One must stand back from the individual items and lessons and evaluate in a global way what kinds of understanding the students expressed. In Chapter Five, items for the teaching experiment were developed from the analysis of the understanding of integers and the operations of integer addition and subtraction within the environment of the neutralization model. These items were placed into a "model of understanding" to indicate the nature of the understanding of each item (physical or emerging mathematical; intuitive, procedural, abstraction or formalization). It is of value at this time to consider what levels of understanding were acquired by the students, and at which levels the difficulties emerged.

Also of interest is to note the general obstacles to the learning of integers which emerged, as well as any obstacles which the neutralization model either overcame or contributed as new ones.

The teaching experiment itself must be evaluated in terms of its effectiveness, its research contributions, and its implications for future research.

8.1 EVALUATION OF LEVELS OF INTEGER UNDERSTANDING

The chart which appears below was developed in Chapter Five, as a means of identifying levels of understanding of integers and the operations of integer addition and subtraction based on the neutralization model as the concrete model for instruction. This chart is repeated again here, in order to evaluate the levels of understanding which the four students displayed. The items in boldface type indicate the problem areas that emerged during the teaching. All other items were understood by all four students, as reported within Chapter Seven, but are not of interest here, except that a comment should be made about the relative ease with which the students constructed understanding of those notions and procedures. Note that item (7) was not necessary during the instruction, as the children spontaneously used zero to represent the neutral element.

PHYSICAL UNDERSTANDING: INTEGERS AS OPPOSITES		
INTUITIVE UNDERSTANDING	PROCEDURAL UNDERSTANDING	LOGICO-PHYSICAL ABSTRACTION
(1) oppositeness (5) neutralization of opposites (6a) neutral element	(6b) formation of neutral element (3) resultant value of group of same kind of chips (11) resultant value of group of chips of both kinds (14) addition of chips of like signs (18) addition of equal amounts of chips of opposite signs (21) addition of chips of unlike signs (25) subtraction of chips of like signs when enough (29) subtraction of chips of equal resultant value (33) **subtraction of chips when not enough** (37) quantitative comparison of chips	(8,9) equivalence classes of neutralization and neutral element (13) **equivalence class of resultant values** (43) **reversibility of addition and subtraction** (45) **equivalent transformations on chips**

EMERGING MATHEMATICAL UNDERSTANDING: INTEGERS AS NUMBERS		
PROCEDURAL UNDERSTANDING	LOGICO-MATHEMATICAL ABSTRACTION	FORMALIZATION
(16) addition of integers with like sign (23) addition of integers with opposite signs (decomposition) (27) subtraction of integers when enough (35) **subtraction of integers when not enough to take away** (38) quantitative comparison of integers (41) positional ordering of integers	(19) addition of opposites neutralize and result in zero (32) subtraction of equal integers results in zero (42) distinction between "larger" integer and "greater than" ordering (44) **reversibility of operations on integers** (46) **equivalence of operations on integers** (47) **awareness of results of operations on integers**	(2,4,12) integer notation of raised sign with numeral (7) notation of NE as the neutral element (10) +n and -n are opposite integers, where n is a whole number (15,22) integer notation for additions (17) integer addition rule for like signs (20) zero as the neutral element (24) rule for integer addition of unlike signs (26,30,34) integer notation for subtractions (28) subtraction rule for integers when enough to take away (31) identification of zero as "nothing" (36) **subtraction rule for integers when not enough to take away** (39) zero as origin (40) integers as indicators of position

PHYSICAL LEVEL OF UNDERSTANDING

Item (13), **equivalence class of resultant values**, was intuitively understood in one direction, that of forming, isolating (and ignoring) the neutral elements to find the resultant value of any group containing both kinds of chips. However, the inverse operation, that of (a) changing the representation of a positive or negative number by adding some neutrals to it, (b) recognizing that the mixture was one group which represented the integer, and then (c) separating the positives and negatives into two separate sub-groups which still represented the integer (for subtractions) was not so well understood. Perhaps a discussion of when each kind of transformation was advantageous would have aided understanding of the true reversibility of the equivalence class of resultant value.

The only difficulty with the procedure for this reversibility was experienced by both Connie and Jason who, when asked to make a few equivalent representations of a particular integer, would initially make only two: one "canonical" representation with chips of one kind, and one representation containing neutrals. Both of these students, confronted with a chip subtraction that did not contain enough to take away, accepted the notion of changing the representation of the initial group. However, they did not immediately see without intervention that adding neutrals to this group had provided a method of solution for the subtraction, since they saw the new group as two separate groups (the neutrals, and the original quantity) rather than a new mixture, containing more of each kind than before, that was equivalent to the old one.

Item (33), **subtraction of chips when not enough to take away**, was understood conceptually, i.e. the situation was recognized as a subtraction which would need the addition of neutral elements in order to provide enough quantity to subtract. Each student experienced a certain degree of difficulty in being able to decide how many neutral elements to add, and in fact, occasionally added only chips of the kind to be subtracted. Connie did not seem to recognize that it was ideal to add the least amount of neutral elements, but continued to add a random amount or to be distracted by the quantity represented by the subtrahend. The author was not careful enough to completely separate the two types of subtractions which fall into this category: those of differing signs where the number of neutrals to add equals the quantity of the subtrahend, and those of the same sign, where some of the kind to be removed are already present. The oversight was based on the assumption that the students would be able to group them both into the category of needing neutrals added, and would be able to evaluate for each particular task how many were necessary. It had not been anticipated that the students would try to generalize about the group addition (such as: add as many as the last quantity), and this contributed to the confusion surrounding the number of neutral elements to contribute.

The question must be raised here about how the students viewed the addition of the neutral elements. This was the first notion (that we can change the representation of the minuend to our advantage, and not just because it was requested as an exercise) that was not an "intuitive" one in the instruction, and it appears that it may have been interpreted as a procedural "trick" that allowed "take away" subtraction, but that did not really hold any meaning or understanding for the students. Clearly not enough time was spent on this notion with the physical model, as the same problems emerged in the realm of integers.

Item (43), **reversibility of addition and subtraction**, was not explored using chips, due to an oversight on the part of the author.

Item (45), **equivalent transformations on chips**, was not explored using physical chips, but by using pictures of chips, as in the following example where the students were asked to return to the value of the initial state by a second transformation.

initial state: ⊕⊕⊕⊕ ⊖⊖ subtract 2 negatives ⊕⊕⊕⊕

For this task all students stated that one could either add 2 negatives, or subtract two positives, but usually they had to be asked if they could find a second transformation. Not enough time was spent at this level, due to the neglect of the author, as there was no writing of the related integer sentences ($^+4 + ^-2 = ^+2$ and $^+4 - ^+2 = ^+2$) for any of these tasks, and the tasks were only three in number. Although the students appeared to have no difficulty with this notion of equivalent transformations at the physical level, it is felt that it was not developed well enough for them to be able to generalize about it.

<u>EMERGING MATHEMATICAL LEVEL OF UNDERSTANDING</u>

Item (35), **subtraction of integers when not enough to take away**, was not well understood due to the lack of time spent on this notion at the physical level. The students had not been able to generalize with the chips and were therefore unable to construct a solid mental model based on the chips which could assist them with these integer subtractions. Steven and Robert, who used the expanded number sentences for these subtractions, used them only as procedural manipulations, although referring to notions based on the manipulations with the chips. Jason was the only student who constructed an understanding of these subtractions, but it was based on the positional representation of integers, with operations as shifts on the number line. As stated above, the two cases of this type of subtraction were not adequately dealt with as <u>separate</u> cases.

Item (36), **subtraction rule for integers when not enough to take away**, was premature. Rules formulated were still at the level of the physical model. Jason made rules based on the number line (subtract to the zero with what you have, then cross over the zero when the signs are the same; subtract farther to the left if it's a negative minus a positive, because you are taking more away so it will become more negative; and move farther to the right if it's a positive minus a negative because you are making it less negative). The others were able to say that you should "add neutrals" to increase the quantity, and subtract the desired amount, but this rule was not adequate enough to supply them with a procedure for finding the number of neutrals necessary (if the signs are the same, subtract; if the signs are opposite, add), nor to focus on the outcomes:

(a) when both signs are the same, the answer to the task is the part of the neutral element whose sign is opposite to that of the subtrahend: $^{+}6 - {}^{+}11 = {}^{-}5$ because 5 neutrals are needed, and the positives are cancelled.

(b) when the signs are different, the answer to the task will be the sum of the leading integer and the part of the neutral element whose sign is opposite to that of the subtrahend: $^{+}8 - {}^{-}3 = {}^{+}11$ because 3 neutrals are needed, and the negatives are cancelled.

It is not until these outcomes are reflected on that the student can construct a more concise rule: you could just add the opposite of the subtrahend to get the same outcome.

Item (44), **reversibility of operations on integers**, was not attempted at the physical level, and the tasks at the mathematical level had not been well thought out.

Item (45), **equivalence of operations on integers**, was noticed by all four students, but there was not enough time spent on this at the physical level (nor perhaps at the mathematical level) for the student to realize the power of this notion, and to be able to discriminate when best to apply it.

Item (47), **awareness of results of operations on integers**, was not developed by the students due to the lack of time spent on these operations. It may take days or weeks of exposure to notice such things as: whenever I subtract integers of opposite signs, I increase the <u>quantity</u> of the leading integer ($^{+}8 - {}^{-}3$ increases the number of positives, and $^{-}4 - {}^{+}7$ increases the number of negatives). It was unrealistic to expect that the students would notice in such a short time that addition and subtraction both have the effect of either an increase or a decrease in the original integer, even if they were aware of the equivalence of operations. A fuller list of results of operations could be made, but it is felt that not many items on the list would be realized in such a short time of instruction without opportunity for more reflection.

There were no estimation tasks, and in retrospect it is felt that the inclusion of this kind of task for both addition and subtraction may have been beneficial in confronting the students with the awareness of results of operations.

INTEGERS AS A NUMBER SYSTEM

The discussion in Chapter Three led to the identification of the criteria for integers as a number system: linearly ordered, presence of both an identity element for addition and an inverse element for addition, equivalence classes of integers, and closure under addition and subtraction. These criteria appear in the above model of understanding in the following items:

linearly ordered: item 41

identity element for addition (zero): items 6a and 6b for the physical model, item 20 for integers

inverse element for addition (opposites): items 5 and 18 for the physical model, items 10 and 19 for integers

equivalence classes of integers: item 13 for the physical model, and items 12 and 13 for integers.

closure: this is not a specific item in the chart, but is dealt with implicitly if it is recognized that no subtractions are impossible when in an integer environment.

None of these notions about integers were difficult for the students to understand, and a mental model for integers based on the neutralization model should thus provide a good foundation for further study of integers as a mathematical group of numbers.

8.2 OBSTACLES TO THE UNDERSTANDING OF INTEGERS

Historically, some obstacles which prevented mathematicians from accepting negative numbers were that of the divided number line, the quantitative notion of number (so nothing could be "below" zero), and the lack of meaning for an isolated negative number (refer to section 3.2).

Before instruction, all three boys gave an indication of experiencing the first of these obstacles - that of the divided number line (which could be also thought of as seeing two separate number systems, positive and negative). This was shown in the way that they had separate rules for ordering each section of the line, and in the way that for Jason, the right side of zero was the "regular" known numbers, and the left side was the negative numbers. They had also constructed some kinds of rules based on the oppositeness of "positive" and "negative" that gave them reasoning for using opposite procedures when the number being operated with was negative. The neutralization model itself could not deal directly with the number line, however there were certain features of this model which contributed to the acceptance of integers as one set of numbers:

(a) its equal treatment of positive and negative numbers, with the quantitative concrete embodiments of the chips

(b) the classifications of procedures for operations were not based on positive or negative, but on the situation which could occur no matter what the signs of the numbers (or chips) were (not enough to subtract, addition results in a mixture, etc.)

Before instruction, Jason, Connie and Marilyn all commented on negative numbers not "existing" for them in the context of operations. This may have also been true for Robert, who had always re-arranged the addition of a negative to a whole number so that he could add the whole number to the negative. All but 13 of the 97 students pre-tested gave meaning for an isolated negative number (in a positional number line context, or as the result of subtraction, which does not mean that they existed as "number" for them). The neutralization model, with its embodiment of integers as quantities, quickly eliminated these two obstacles for these students, who demonstrated that integers existed as mental objects for them, which could be decomposed, added and subtracted.

Another obstacle which emerged was the proliferation of signs for integer operations, which proved to be a distractor for each student at various times. When given a series of mixed tasks after instruction on addition and subtraction, there were times when the focus of the student was not on the sign of the operation, but on the signs of the numbers (for example, for $^{-}n - {}^{+}n$, a student might have explained that the signs are opposite (or the numbers are opposites), so this would be a neutralization task. It seems that this obstacle might emerge no matter which concrete model was used for instruction.

An obstacle which emerged directly from the neutralization model was that the procedure for some additions included "removal" of neutrals, and that the procedure for some subtractions included "addition" of neutrals. Sometimes when given a mixture of tasks, the first response of the student was to try to "add neutrals" for the addition tasks, and to "neutralize" or "remove neutrals" for the subtraction tasks.

8.3 THE SUBTRACTION RULE

The question arises - why not just give the subtraction rule for those subtractions which give the most difficulty, as many authors recommended? Although these four students rarely used the rule on their own, it seemed to make more sense for them when the sign of the operation was followed by the same sign on the number (i.e. $m + {}^{+}n$, $m - {}^{-}n$). This would transform ${}^{+}L - {}^{-}S$ and ${}^{+}S - {}^{-}L$ into trivial additions, and ${}^{-}S - {}^{-}L$ into a neutralizing addition, and this limited use of the subtraction rule is justified, as it can be backed up with the concrete model of chip subtraction. It would be interesting to interview or test students who have learned to subtract with this rule to see if it is more successfully used in this limited framework.

Although many authors felt that this rule would be "obvious" to the students, indeed it was not, except within the specific task of finding two ways to achieve a target integer from a given initial integer. It would take an extended exposure to integer operations to develop this notion on their own, and even then, it would be used as a powerful short-cut, rather than a "rule", as it is not needed for all cases, and as it can also be used to make some additions into trivial ones.

8.4 A COMMENT ON CANCELLATION

Although the use of the expanded sentences for addition and subtraction has now been brought into question, it may be said here that it had been expected that the necessary cancellation might not have been obvious to the student. In this regard it was felt that the cancellation procedure on whole numbers needed to be brought to the attention of the students before it was encountered with integers. When cancellation of whole numbers was presented within the context of neutralization, some of the original problems were overcome. As it turned out, all of the work on cancellation could have been omitted if these expanded sentences had been dropped from the instruction. However, since it was included, it must be stated that not only do students often not see the possibility of cancellation with whole numbers, but there were many difficulties stemming from the proliferation of signs with the cancellation procedures for integers, and that students tended to focus on either the signs for the operations or the signs of the operations, but not both at once. These problems were not so frequent when put into the operational modelling of addition and subtraction procedures, since the object of writing the expanded sentence was to cancel either neutrals (in an addition sentence), or to cancel a subtraction with an addition (in a subtraction sentence), and this focus proved to be adequate to overcome most integer cancelling difficulties (tried with only two students).

8.5 EVALUATION OF THE TEACHING EXPERIMENT

This teaching experiment was designed to present notions about integers and about integer addition and subtraction within the context of a concrete model which would give meaning to integers and to these operations. Previous surveys had shown consistently lower success rates on subtraction tasks, and it was hoped that this model would provide more even rates of success for both additions and subtractions.

First of all, this teaching experiment must be considered as a pilot study only, as refinements are needed. Within the given time frame of five lessons, the goals were too ambitious. The first two lessons (introductory integer notions and additions) presented no difficulties for the students, but more exposure to additions of chips and integers would have been beneficial so that the students could construct generalizations about the results of additions before subtractions were introduced.

Subtraction, which was the content of lesson 3, should have been introduced more slowly, with much more subtractions at the concrete level, with sentences to document the subtractions. It appears that children may not be able to make the jump between the concrete model and the mental model so easily (as evidenced by a student like Connie who had no difficulty with the concept of subtraction at the concrete level, but who was often not able to think of a "take away" subtraction at the integer level). The two types of "subtractions when not enough to remove" should have been more clearly separated more than they were, so that the second kind would have provided a new conflict situation leading to resolution of the particular problem and a new strategy. It is recommended that no time frame be put on instruction for this operation until further individualized experiments can be conducted in order to determine the amount of exposure the student would have to have in order to fully understand integer subtraction.

It is certain that, using this model, additions are easier than subtractions, although the concern that Bélanger (1984) had about the notion of changing the representation of the minuend was unfounded, since it made sense to them that there had to be a change to allow them to have enough to take away. However, Galbraith's (1974) concern about what form the representation would take was a valid one, and perhaps the student does have to be in the Formal Operational stage before he can discriminate exactly what to add. We spend a lot of time teaching the elementary school student how to "borrow a unit from the next larger column" for subtraction, and even then, this task is not without difficulty, so it is only fair to treat this step in particular (when we do not always "borrow" the same amount) as deserving of more time and practise.

The expanded addition and subtraction sentences were of no benet. for constructing understanding of these operations on integers, although Steven and Robert were sometimes able to use them as a procedure to find correct answers. These sentences often brought more confusion to the situation rather than clarification, and they should be eliminated from the design of the experiment.

The number line was used as a model for ordering. It might have been possible to use the ideas of "more positive" and "more negative" as criteria for ordering, but the multi-representations of each integer may present an obstacle here.

Lesson Five (generalizations) was premature. The students had not spent enough time reflecting on addition and subtraction as individual processes before they were asked to reflect on them as related operations. Although some preliminary abstractions were beginning to be expressed by the students, this lesson could have been considered to be beyond the scope of the original problem. The short time spent on both addition and subtraction led to a concentration on the "skill" of performing these operations, rather than on building understanding of what was happening in the operations.

8.6 CONTRIBUTIONS TO RESEARCH

Even though the research conducted for this thesis is considered to be of an exploratory nature, some developments are of benefit to the research community.

The conceptual analysis developed in Chapter Five for the understanding of integers and the operations of integer addition and subtraction, translated into tasks and levels of understanding using a model which had been developed by Herscovics & Bergeron (1988), appears to be the first attempt of this nature with respect to the neutralization model. Since it had been discovered that many authors who recommended this model had not exploited the concrete model to its full potential, or had not suggested how to move from the physical objects of chips etc., to the mathematical (mental) ones of integers, it is hoped that the conceptual analysis will prove to be useful to educators as a complete framework for teaching. It can also be used to analyze the levels of understanding that their students construct.

Herscovics & Linchevski (1991-a, 1991-b) in their research with grade 7 students in Montreal, found that when asked to solve a string of operations, or to cancel before solving, some students "jumped off with the posterior operation", and some "detached from the minus sign" and grouped before subtracting. This present study confirms that these two phenomena are indeed experienced by students at this grade level, and suggests that one of the difficulties is with the misconception that since the leading number has no sign in front of it, it is not seen as an "addition", and becomes associated with sign following it (the posterior operation).

In addition to these difficulties observed with whole number cancellations in strings of operations, this study revealed difficulties with cancellations of integers in strings. Students tended to concentrate on one type of sign only, either that of the operations or that of the integers, indicating that integer cancellations and operations in strings are more demanding than those of whole numbers.

8.7 IMPLICATIONS FOR FUTURE RESEARCH

As was mentioned earlier, this teaching experiment can be considered as an exploratory study, and recommendations for future experiments based on refinements can be made.

It is left to be determined whether, given more exposure to addition and subtraction, and provided with more time and activities to reflect on these operations, subtraction difficulties would be overcome. These activities should be more fully developed than the ones in this experiment. The number of times necessary to meet with a student must not be pre-determined if this is to be explored fully, and if skills are not to overshadow understanding.

As mentioned, ordering was not taught using the neutralization model. It would be of benefit to examine the notions of ordering more thoroughly within the context of the neutralization model to see if there is some way (without being contrived) to intuitively bring about understanding of linear ordering.

Multiplication and division were not within the scope of this thesis, and as these seem to not be well modelled within the integer domain by other models, it would be worth investigating if these could be represented meaningfully within the context of the neutralization model.

As mentioned in Chapter One, a couple of small-scale studies have tried to evaluate why success rates for subtraction are lower than those for addition, and it was found that students had a mental model to substantiate their additions, but did not for the subtractions. It would be of interest to design a better assessment of student understanding for students who have already had integer instruction (students in grades 8 through 11) to evaluate their level of understanding according to the Herscovics-Bergeron model, in a format of individual interviews. What awareness of integer operations have these students developed simply from amount of exposure to integers both in arithmetic and in algebraic environments?

At times during this experiment, the calculator was used to verify results of operations, especially when a chip procedure gave one result, and the student's reasoning had given another. These four children had demonstrated trust in calculator results (sometimes after they had verified their keystrokes) in whole number tasks, and it would be useful to conduct a literature review of any research that has been undertaken to determine if the calculator is viewed as an authority, or as a "black box".

It is thought that the conceptual analysis developed in Chapter Five could be adapted in such a way as to be used to analyze student's understanding based on the number line rather than on neutralization. Most items at the emerging mathematical level could remain unchanged, while items at the physical level would use the number line as the concrete model. This seems to be needed, since the number line is the most common model used in teaching, and since few studies have focused on the understanding of integers which students have constructed using this model. It would also serve to evaluate how well students who have been taught with the number line understand the physical model itself, which had been a concern of some of the authors.

Research into children's ideas about cancellation is an area where work is needed, especially investigations into the following: understanding that the answer to a string is preserved after cancellation; understanding of which sign to "cancel" with each number; conceptions about the order in which the numbers appear in the string (subtraction before or after addition); understandings or misunderstandings about cancellation of the leading number; and affect of distance between numbers which can be cancelled (i.e. juxtaposed or separated).

BIBLIOGRAPHY

Artzt, A. F. & Newman, C. M. (1991). Equivalence: A Unifying Concept. Mathematics Teacher, 84(2), 128-132.

Ashlock, R. B. & West, T. A. (1967). Physical Representations for Signed-Number Operations. The Arithmetic Teacher, 14(7), 549-554.

Aze, I. (1989). Negatives - Are They for Little Ones? Micromath, 5(1), 15-17.

Baily, C.A.R., Barton, A., Blakey, J.C., Brighton, D.R., Bromby, H., Davey, P.C. (1974). Mathematics Eleven to Sixteen, A Report Prepared for the Mathematical Association, G. Bell and Sons Ltd., England, pp. 46-53.

Bartolini, P. (1976). Addition and Subtraction of Directed Numbers. Mathematics Teaching, 74, 34-35.

Battista, M. T. (1983). A Complete Model for Operations on Integers. Arithmetic Teacher, 30(9), 26-31.

Bélanger, G. (1984). Une Approche Intuitive pour l'Enseignement des Entiers Relatifs. Instantanés Mathématiques, 21(2), 5-12.

Bélisle, J.-G. (1989). Une Introduction aux Entiers Relatifs. Instantanés Mathématiques, 25(5), 19-23.

Bell, A. (1982). Looking at Children and Directed Numbers. Mathematics Teaching, 100, 66-72.

Bell, A. (1983). Direct Numbers and the Bottom Up Curriculum. Mathematics Teaching, 102, 28-32.

Bell, A. W., Costello, J., & Kuchemann, D. (1983). Research on Learning and Teaching: A Review of Research in Mathematical Education, Part A. Nfer-Nelson, Britain, pp. 128-131.

Bennett, A. B. Jr., & Musser, G. L. (1976). A Concrete Approach to Addition and Subtraction. The Arithmetic Teacher, 23(5), 332-336.

Boyer, C. B. (1968). A History of Mathematics. John Wiley & Sons Inc., New York. pg. 224.

Brien, M. (1979). untitled, final project to fulfill the requirements for math 643, Concordia University, unpublished

Brown, M. (1986). Outdoor Negatives. Mathematics Teaching, 114, 14-15.

Carpenter, T., Coburn, T., Reys, R. and Wilson, J. (1978). Results from the FIRST MATHEMATICS ASSESSMENT of the National Assessment of Educational Progress, NCTM, pg. 51.

Carpenter, T., Corbitt, M. K., Kepner, H., Lindquist, M. and Reys, R. (1981). Results from the SECOND MATHEMATICS ASSESSMENT of the National Assessment of Educational Progress, NCTM, pp. 53-54.

Carr, K. & Katterns, B. (1984). Does the Number Line Help? Mathematics in School, 13(4), 30-34.

Chalouh - Hertzman, L. (1979) untitled, Seminar on Teaching Mathematics, Project 1, Concordia University, unpublished course project in MTM program.

Chang, L. (1985). Multiple Methods of Teaching the Addition and Subtraction of Integers. Arithmetic Teacher, 33(4), 14-19.

Chang, L. & Ruzicka, J. (1985). Second International Mathematics Study: United States Technical Report 1, Stipes Publishing Company, Illinois, pg. 82.

Chilvers, P. (1985). A Consistent Model for Operations on Directed Numbers. Mathematics in School, 14(1), 26-28.

Cochran, B. S. (1966). Children Use Signed Numbers. <u>Arithmetic Teacher</u>, <u>13</u>(7), 587-588.

Cohen, L. S. (1965). A Rationale in Working with Signed Numbers. <u>The Arithmetic Teacher</u>, <u>12</u>(7), 563-567.

<u>Computation and Structure</u>, (1969). Nuffield Mathematics Project, Newgate Press Ltd. London, England. pp. 22-37.

Coon, L. H. (1966). Number Line Multiplication for Negative Numbers. <u>The Arithmetic Teacher</u>, <u>13</u>(3), 213-217.

Cotter, S. (1969). Charged Particles: a Model for Teaching Operations with Directed Numbers. <u>Arithmetic Teacher</u>, <u>16</u>(5), 349-353.

Crosswhite, F. J., Dossey, J. A., Swafford, J. O., McKnight, C. C., Cooney, T. J., Downs. (1986). <u>Second International Mathematics Study</u>: Detailed Report for the United States, Stipes Publishing Company, Illinois, pp. 81-82.

Dirks, M. K. (1984). The Integer Abacus. <u>Arithmetic Teacher</u>, <u>31</u>(7), 50-54.

Eastwood, M. (1983). More Models for Directed Number. <u>Mathematics in School</u>, <u>12</u>(3), 34-35.

Ernest, P. (1985). The Number Line as a Teaching Aid. <u>Educational Studies in Mathematics</u>, <u>16</u>(4), 411-424.

Fischbein, E. (1977). Image and Concept in Learning Mathematics. <u>Educational Studies in Mathematics</u>, <u>8</u>, 153-165.

Fremont, H. (1966). Pipe Cleaners and Loops - Discovering how to Add and Subtract Directed Numbers. <u>The Arithmetic Teacher</u>, <u>13</u>(7), 568-572.

Freudenthal, H. (1983). Didactical Phenomenology of Mathematical Structures. Chapter 15. D. Reidel Publishing Co. Holland. pp. 432-460.

Fuson, K. (1984). More Complexities in Subtraction. Journal for Research in Mathematics Education, 15(3), 214-225.

Galbraith, M. J. (1974). Negative Numbers. International Journal of Mathematical Education in Science and Technology, 5, 83-90.

Goncalves, M. J. P. (1982). Methods of Teaching Mathematics. Final project for Math 643, Concordia University, unpublished.

Gibbs, R. A. (1977). Holes and Plugs. Mathematics Teaching, 81, 19-21.

Glaeser, G. (1981). Épistémologie des Nombres Relatifs. Recherches en Didactique des Mathématiques, 2(3), 303-346.

Grady, M. B. T. (1978). A Manipulative Aid for Adding and Subtracting Integers. Arithmetic Teacher, 26(3), 40.

Greeno, J. G. (1991). Number Sense as Situated Knowing in a Conceptual Domain. Journal for Research in Mathematics Education, 22(3), 170-218.

Grignon, J. (1989). Un Coup d'Oeil sur la Soustraction et la Droite Numérique. Instantanés Mathématiques, 26(2), 29-30.

Hall, W. H. (1974). These Three: Minus, Negative, Opposite. The Arithmetic Teacher, 21(8), 712-713.

Haner, W. W. (1947). Teaching the Subtraction of Signed Numbers. School Science and Mathematics, 47, 656-658.

Harvey, A. A., & Cunningham, V. (1980). The Integers Test. Project submitted in partial fulfillment of the requirements for the degree of MTM, Concordia University, (unpublished).

Hefendehl-Hebeker, L. (1991). Negative Numbers: Obstacles in their Evolution from Intuitive to Intellectual Constructs. For the Learning of Mathematics, 11(1), 26-32.

Herscovics, N. & Bergeron, J. C. (1988). An Extended Model of Understanding. Proceedings of Psychology of Mathematics Education - North America - 10, Behr, M. J., Lacampagne, C. B. & Wheeler, M. M., (editors), Northern Illinois University, DeKalb, Illinois, 15-22.

Herscovics N. & Linchevski, L. (1991-a). Pre-algebraic Thinking: Range of Equations and Informal Solution Processes used by Seventh Graders prior to any Instruction. Proceedings of Psychology of Mathematics Education - 15 (volume 2), Furinghetti, F. (editor), Genova, Italy, 173-180.

Herscovics, N. & Linchevski, L. (1991-b). Crossing the Didactic Cut in Algebra: Grouping Like Terms in an Equation. Proceedings of Psychology of Mathematics Education - North America - 13 (volume 2), Underhill, R. G., (editor), Blacksburg, Virginia, 196-202.

Human, P. & Murray, H. (1987). Non-Concrete Approaches to Integer Arithmetic. Proceedings of the Ninth Annual Conference for the Psychology of Mathematics Education, Vol. 1, Leen Streefland (editor), State University of Utrecht, the Netherlands, 437-443.

Hursh, R. E. (1966). The Number Line in the Junior High School. The Arithmetic Teacher, 13(7), 553-555.

Integers and Combining Like Terms in Algebra, (1986). Manitoba Education Curriculum Development and Implementation, Winnipeg, Manitoba.

Integers and Computations with Integers, (1990). Project Focus, School of Education, Boston University, Boston (revised 1991).

Janvier, C. (1983). The Understanding of Directed Numbers. Proceedings of Psychology of Mathematics Education - North America - 5, Vol.2, Bergeron, J. & Herscovics, N. (editors), Montreal, 295-301.

Janvier, C. (1985). Comparison of Models Aimed at Teaching Signed Integers. Proceedings of the Ninth Annual Conference for the Psychology of Mathematics Education, Vol. 1, Leen Streefland (editor), State University of Utrecht, the Netherlands, 135-140.

Jencks, S. M. & Peck, D. M. (1977). Hot and Cold Cubes. The Arithmetic Teacher, 24(1), 70-71.

Kohn, J. B. (1978). A Physical Model for Operations with Integers. Mathematics Teacher, 71(9), 734-736.

Kuchemann, D. (1975). CSMS items and percentage results. Personal communication to Nicolas Herscovics.

Kuchemann, D. (1980). Children's Understanding of Integers. Mathematics in School, 9(2), 31-32.

Kuchemann, D. (1981). Positive and Negative Numbers. published in Children's Understanding of Mathematics: 11 - 16, K. Hart, editor, pp. 82-87.

Leddy, T. (1977). Mis-Directed Numbers. Mathematics Teaching, 78, 26-29.

Liebeck, P. (1990). Scores and Forfeits - An Intuitive Model for Integer Arithmetic, Educational Studies in Mathematics, 21, 221-239.

Lindquist, M. M. (editor) (1989). Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress, NCTM, pg. 88.

Luckow, A. (1982). Adding and Subtracting Integers. Pilot Study in partial fulfillment of Mathematics 643, Concordia University, unpublished.

Luth, L. M. (1967). A Model for Arithmetic of Signed Numbers. The Arithmetic Teacher, 14(3), 220-222.

Lytle P. and Avraam, J. (1990). Integer Addition and Subtraction. (paper written for course requirements in MTM program).

Lytle, P. and Avraam, J. (1991). Teaching Experiment to Overcome the Learning Difficulty of Integer Subtraction. (paper written for course requirements in MTM program).

Magnuson, R. C. (1966). Signed Numbers. Arithmetic Teacher, 13(7), 573-575.

McAuley, J. (1990). "Please sir, I didn't do nothin". Mathematics in School, 19(1), 45-47.

Mertz, B. (1979). Negative Numbers at the Wrong Time. Mathematics Teaching, 89, 5.

Mukhopadhyay, S., Resnick, L. B. & Schauble, L. (1990). Social Sense-Making in Mathematics; Children's Ideas of Negative Numbers. Proceedings of Fourteenth Psychology of Mathematics Education Conference (Vol. 3), Booker, G., Cobb, P. & Mendicuti, T. N. (editors), Mexico. 281-288.

Murray, J. C. (1985). Children's Informal Conceptions of Integer Arithmetic. Proceedings of the Ninth Annual Conference for the Psychology of Mathematics Education, Vol. 1, Leen Streefland (editor), State University of Utrecht, the Netherlands, 147-153.

Nabbie, H. (1979). Addition and Subtraction of Integers. unpublished course project in MTM program, Concordia University.

Nantais, N. & Herscovics, N. (1989). Epistemological Analysis of Early Multiplication. Proceedings of Thirteenth Annual Psychology of Mathematics Education conference, Vol. 3, Vergnaud, G., Rogalski, J & Artique, M (editors), Paris, France, 18-24.

Peled, I. (1991). Levels of Knowledge About Signed Numbers: Effects of Age and Ability. Proceedings of Fifteenth Annual Psychology of Mathematics Education conference, Vol. 3, Fulvia Furinghetti (editor), Assisi, italy, 145-152.

Peled, I., Mukhopadhyay, S. & Resnick, L. (1989). Formal and Informal Sources of Mental Models for Negative Numbers. Proceedings of Psychology of Mathematics Education - 13, Vol. 3, Vergnaud, G., Rogalski, J. & Artigue, M. (editors), Paris, 106-110.

Pycior, H. M. (1984). Internalism, Externalism, and Beyond: 19th-Century British Algebra. HIstoria Mathematica, 11, 424-441.

Rabin, E. (1979). A Case Study on the Teaching of Integers. paper submitted in partial fulfillment of the requirements for the degree of MTM, Concordia University, unpublished.

Rathmell, E. C. (1980). Concepts of the Fundamental Operations: Results and Implications from National Assessment. Arithmetic Teacher, 28(3), 34-37.

Relative Integers, (1986). Curriculum guide, Elementary School Mathematics, Booklet D., Gouvernement du Québec, Ministère de l'Éducation, Québec.

Rowland, T. (1982). Teaching Directed Numbers: An Experiment. Mathematics in School, 11(1), 24-27.

Schubring, G. (1986). Ruptures dans le Statut Mathématique des Nombres Négatifs. petit x, 12, 5-32.

Semadeni, Z. (1984). A Principle of Concretization Permanence for the Formation of Arithmetical Concepts. Educational Studies in Mathematics, 15(4), 379-394.

Shawyer, B. (1985). Positive, Negative, Plus, Minus, Add, Subtract. Ontario Mathematics Gazette, 23(3), 34-35.

Sherzer, L. (1969). Adding Integers using only the Concepts of one-to-one Correspondence and Counting. The Arithmetic Teacher, 16(5), 360-362.

Smith, D. E. (1925). History of Mathematics Volume 2. Ginn & Brothers, Boston U.S.A. pg. 257.

Tobias, R. K., Anderson, K., Cahill, M. J., Carter, D. A., Duffin, J. M., Goodwin, J. A. (1982). Mathematics for the Middle Years: A Spiral Development: Part 6. Mathematics in School, 11(2), March, 16-18; 11(3), May, 14-18; 11(4), Sept., 20-24; 11(5), Nov., 5-11.

Werner, Sister M. (1973). The Case for a More Universal Number-line Model of Subtraction. The Arithmetic Teacher, 20(1), 61-64.

Whimbey, A. & Lochhead, J. (1981). Boxes of Charged Particles for Teaching Signed Number Rules. Focus on Learning Problems in Mathematics, 3(4), 1-4.

APPENDIX A

RESULTS ON ASSESSMENT TESTS

Results (% Success) on Various Assessment Studies on Integer Items

ADDITION ITEMS

ITEM FORMAT	AGE 13	AGE 14	AGE 15	AGE 17	GRADE 7	GRADE 8	GRADE 9
pos. + pos.							
$^+5 + ^+3$ (H. & C.)						94%	
$^+2 + ^+6$ (L. & A.)	100%	100%	100%				
$^+2 + ^+4$ (CSMS)	64%	60%	63%				
$^+2 + ^+6$ (CSMS)	95%	97%	96%				
Morocco						88%	
$^+ \ + ^+$ (NAEP 2)	90%			93%			
neg. + neg.							
$^-5 + ^-3$ (H. & C.)						86%	
$^-9 + ^-4$ (L. & A.)	90%	96%	94%				
$^-9 + ^-4$ (CSMS)	64%	78%	73%				
$^- \ + ^-$ (NAEP 2)	75%			80%			
$^- \ + ^-$ (APU) 65%							
$^-7 + ^-5$ (S.A. - 1)						61%	74%
$^-5 + ^-3$ (S.A. - 3)					53%		
Morocco						68,62	
$^-5 + ^-9$ (NAEP 1)	66%			78%			
$^+$S + $^-$L							
$^+5 + ^-9$ (H. & C.)						83%	
Morocco						66%	
$^+2 + ^-3$ (L. & A.)	80%	92%	100%				
$^-$S + $^+$L							
$^-9 + ^+11$ (H. & C.)						82%	
Morocco						70%	
-4 + 7 (S.A. - 1)						48%	77%
$^+$L + $^-$S							
$^+7 + ^-4$ (H. & C.)						84%	
$^+8 + ^-4$ (L. & A.)	75%	88%	82%				
$^+8 + ^-4$ (CSMS)	87%	87%	86%				
$8 + ^-5$ (S. A. - 3)					35%		
Morocco						80%	
$10 + ^-3$ (S.A. - 1)						51%	75%

ITEM FORMAT	AGE 13	AGE 14	AGE 15	AGE 17	GRADE 7	GRADE 8	GRADE 9
$^{-}L + {}^{+}S$							
$^{-}8 + {}^{+}3$ (H. & C.)						83%	
$^{-}8 + {}^{+}3$ (L. & A.)	75%	92%	100%				
$^{-}8 + {}^{+}3$ (CSMS)	77%	85%	81%				
Morocco						79%	
$^{-}8 + 3$ (S.A. - 1)						52%	78%
$^{+}n + {}^{-}n$ or $^{-}n + {}^{+}n$							
$^{+}3 + {}^{-}3$ (H. & C.)						87%	
$^{-}7 + {}^{+}7$ (H. & C.)						86%	
$^{+}5 + {}^{-}5$ (L. & A.)	95%	88%	94%				
$7 + {}^{-}7$ (S.A. - 3)					43%		
Morocco						77%	
$^{+}5 + {}^{-}5$ (CSMS)	90%	91%	89%				
OTHER FORMATS							
$^{-}4 + {}^{-}4$ (H. & C.)						71%	
$^{-}n + {}^{-}n$ (Morocco)						60%	
$0 + {}^{-}5$ (H. & C.)						93%	
$^{-}7 + 0$ (H. & C.)						93%	

SUBTRACTION ITEMS

ITEM FORMAT	AGE 13	AGE 14	AGE 15	AGE 17	GRADE 7	GRADE 8	GRADE 9	GRADE 10
^{+}L - ^{+}S								
$^{+}6 - ^{+}3$ (H. & C.)						82%		
Morocco						88%		
$^{+}8 - ^{+}4$ (CSMS)	77%	77%	70%					
^{-}L - ^{-}S								
$^{-}4 - ^{-}2$ (H. & C.)						75%		
Morocco						86%		
$^{-}12 - ^{-}3$ (S.A. -1)						57%	63%	
$^{-}9 - ^{-}4$ (S.A. - 2)							58%	63%
$^{-}12 - ^{-}4$ (S.A.- 3)					56%			
^{+}S - ^{+}L								
$^{+}5 - ^{+}8$ (H. & C.)						77%		
$^{+}6 - ^{+}8$ (L. & A.)	55%	72%	71%					
3 - 8 (S.A. - 1)						48%	69%	
5 - 8 (S.A. - 3)					54%			
^{-}S - ^{-}L								
$^{-}4 - ^{-}8$ (H. & C.)						77%		
$^{-}2 - ^{-}5$ (L. & A.)	45%	60%	65%					
$^{-}6 - ^{-}8$ (IEA 2)						41%		
$^{-}7 - ^{-}10$ (NAEP 2)	27%			54%				
Morocco						74%		
$^{-}5 - ^{-}12$ (S.A. - 1)						34%	55%	
$^{-}5 - ^{-}9$ (S.A. - 2)							40%	48%
$^{-}5 - ^{-}8$ (S.A. -3)					20%			
^{+}L - ^{-}S								
$7 - ^{-}3$ (APU) 50%								
$^{+}8 - ^{-}6$ (L. & A.)	45%	48%	59%					
Morocco						73%		
$8 - ^{-}3$ (S.A. - 1)						17%	46%	
$7 - ^{-}5$ (S.A. - 2)							27%	38%
^{-}L - ^{+}S								
$^{-}6 - ^{+}3$ (L. & A.)	35%	44%	71%					
Morocco						60%		
$^{-}7 - 4$ (S.A. - 1)						21%	50%	
$^{-}7 - 3$ (S.A. - 2)							36%	47%
$^{-}8 - 3$ (S.A. - 3)					19%			

ITEM FORMAT	AGE 13	AGE 14	AGE 15	AGE 17	GRADE 7	GRADE 8	GRADE 9	GRADE 10
$^{+}S - {}^{-}L$								
$^{+}5 - {}^{-}6$ (H. & C.)						66%		
Morocco						72%		
$5 - {}^{-}8$ (S.A. - 3)					3%			
$^{-}S - {}^{+}L$								
$^{-}2 - {}^{+}7$ (H. & C.)						60%		
$^{-}4 - {}^{+}7$ (NAEP 2)	21%			39%				
$^{-}8 - {}^{+}10$ (NAEP 2)	37%			61%				
$^{+}n - {}^{-}n$, $^{-}n - {}^{+}n$								
$^{+}7 - {}^{-}7$ (H. & C.)						66%		
$^{-}5 - {}^{+}5$ (H. & C.)						64%		
Morocco						69,51		
$^{+}5 - {}^{-}5$ (CSMS)	21%	31%	31%					
$^{+}n - {}^{+}n$								
$^{+}2 - {}^{+}2$ (H. & C.)						83%		
Morocco						85%		
$^{+}6 - {}^{+}6$ (L. & A.)	70%	88%	82%					
$^{-}n - {}^{-}n$								
$^{-}8 - {}^{-}8$ (H. & C.)						82%		
$^{-}3 - {}^{-}3$ (L. & A.)	75%	68%	76%					
Morocco						88%		
$^{-}7 - {}^{-}7$ (S.A. - 3)					72%			
Items with zero								
$0 - {}^{-}4$ (H. & C.)						48%		
$0 - {}^{-}n$ (Morocco)						71%		
$0 - {}^{+}6$ (H. & C.)						54%		

Legend:

L = large number relative to other number (using absolute values to determine size)

S = small number relative to other number (using absolute values to determine size)

n always represents a positive integer, ^{-}n a negative integer

References:

H. & C. = Harvey & Cunningham (1980) Note that all integers were bracketed (for example (-2) + (+3))

L. & A. = Lytle & Avraam (1990)

CSMS = Concepts in Secondary Mathematics and Science Study (Kuchemann, 1980,1981)

NAEP = National Assessment of Educational Progress (Carpenter et al.,1978, 1981; Lindquist, 1989)

APU = Assessment Performance Unit (Bell et al., 1983)

S.A. = South Africa (Murray, 1985)

IEA = International Educational Assessment (Chang & Ruzicka, 1985; Crosswhite et al., 1986)

Morocco (Janvier, 1983,1985)

APPENDIX B

TEXTBOOK EVALUATION

CRITERIA FOR TEXTBOOK EVALUATION - TEACHING OF INTEGERS

HISTORICAL OBSTACLES IN THE UNDERSTANDING OF INTEGERS

(1) Negative numbers had no meaning - they arose as solutions to equations such as 10 + $\square$ = 3, and this caused their avoidance for a long time.
Solution: everyday life examples of opposites (up 5, down 5).

(2) Zero was seen as absolute zero , you could not count below this point.
Solution: zero as an origin or reference point.

(3) Procedural rules gave answers to symbolic manipulations, but no one provided a concrete model that would justify the rules (especially the multiplication rule of a negative times a negative equals a positive). This model should be able to demonstrate as many operations as possible.
Solution: provide a concrete model for manipulation of numbers, or operate solely on the level of abstract mathematics.

(4) It was difficult for people to conceive of a continuous number line.
Solution: The integer number line (thermometer, etc.)

(5) The magnitude of a number was more of a focus than the concept of the number.
Solution: See integers as a distinct set of numbers, separate from N (with N isomorphic to the positive integers).

(6) Notation could cause difficulties (recent). Confusion between sign of number and sign of operation, also confusion between positive integers and natural numbers.
Solution: avoid writing the two signs at the same level (e.g. +5 + +3). Also do not omit the + sign for positive integers.

CONCEPTS OF INTEGERS AS A SET OF NUMBERS

(1) Integers have both size and magnitude, and can be ordered in 2 ways.

(2) Creation of a neutral element as integers of equal magnitude cancel each other.

(3) Equivalence classes (this notion may be model specific) - each integer, and zero in particular, may be represented in an infinite number of ways.

(4) Model used for integer operation must be simple enough to not cause obstacles itself, and must maintain concepts of addition and subtraction that exist in N.

(5) Reflection on results of operations are necessary in order to establish some of the properties in Z. (such as : all subtractions are now possible; addition does not always make larger; there are two ways to obtain an answer; every number has an opposite, etc.)

(6) Exercises in the text must ultimately lead the student away from the model, so must contain some questions with large numbers, and some questions which do not refer to a particular model.

(7) Since subtraction has proven to be a difficult operation to perform, there must be adequate explanation and sufficient practise of this operation.

TOPIC - INTEGERS   GRADE 7 (introduction, addition, subtraction)

PART ONE - INTRODUCTION TO INTEGERS : NOTIONS

[1] Integers are a set of numbers, different from N (pre-operational)

Integers must be presented as a set of numbers which are either positive or negative in nature, or neutral (zero). Positive integers should not be confused with whole numbers, since a positive number reflects a position with respect to zero.

TEXT NUMBER

	1	2	3	4	5	6	7	8	9	10
explicit notion	•		•	•	•	•		•		
implicit notion										
not at all		•					•		•	•

[2] Integer Notation

The notation used to indicate integers should not cause any confusion between the operations of addition and subtraction and the sign of the number. Positive integers should be distinctive from whole numbers. A <u>consistent</u> system is necessary. The best notation is either that of superscripts for positive and negative, for example $^{+}7$ and $^{-}2$, or brackets around the integer, for example (+6) and (-4).

	1	2	3	4	5	6	7	8	9	10
super for neg. only				•				•		
super for pos. + neg.	•			•			•			
brackets for neg. only									•	•
brackets for pos and neg		•	•	•	•	•				
brackets and super				•						

The notation in text 4 is very confusing, changing forms with no explanation. Texts 3, 5 and 9 only used brackets if operations were being performed, otherwise the number had a lowered sign (as in -9 and +7).

[3] Meaning attached to integers

Are integers treated as abstract numbers, or is there an attempt to see them as expressions of situations from everyday life (temperatures, debts and profits, stock market situations, elevator floors, etc.)? Problem: in real life, one does not use integer notation to express most of these situations

All 10 textbooks interpreted integers in some fashion related to everyday life. None used the solution to an equation approach.

[4] Notion of zero as origin

In N, zero is "absolute zero" - you can go no lower as you count down. In Z, however, zero represents a reference point, where anything above zero is positive, and anything below zero is negative.

	1	2	3	4	5	6	7	8	9	10
explicit notion				•		•				
implicit notion	•		•				•			•
not at all		•			•			•	•	

The results of this presentation of this notion are surprising - since this is a new concept for the students, and one that has proven to be difficult.

[5] The notion of an unbroken number line

This notion has been universally accepted (and has applications such as the thermometer scale), so was not expected to be a problem.

All texts reviewed showed a continuous number line.

[6] Notion that integers have both size and magnitude

The magnitude (distance from zero) of $^{+}10$ and $^{-}10$ is the same, but $^{+}10$ is <u>larger than</u> $^{-}10$ (i.e. is more positive), just as $^{-}4$ is more positive than $^{-}15$, and $^{+}13$ is colder than $^{+}25$.

	1	2	3	4	5	6	7	8	9	10
explicit notion	•					•				
implicit notion				•						
not at all		•	•		•		•	•	•	•

Again, these results are surprising, since this is a new feature of numbers, unique to Z.

[7] Procedure of Ordering by Size

The number line may be used as it is in N, where numbers are placed on the line, and the number on the right is the larger number. Thus $^{-}3 > ^{-}20$.

	1	2	3	4	5	6	7	8	9	10
use number line as in N	•		•	•	•	•	•	•	•	•
another method		•						•		

Text 2 gave absolutely no criteria for ordering by size, and 8 gave real life examples as well as the number line.

[8] Ordering by Magnitude (Absolute Value)

This procedure is needed for addition of integers of different signs, where the "larger" number and the "smaller" number must be identified, and this identification is based solely on magnitude.

	1	2	3	4	5	6	7	8	9	10
explicit notion				•					•	
implicit notion										
not at all	•	•	•		•	•	•	•		•

Text 4 (a french text) calls this notion "dominance", and text 9 calls it "absolute value", but does not introduce it until very late. The other texts avoid this notion.

[9] Integers as Opposites

Positive and negative are opposites, and integers are used to designate opposite notions. This should be expressed by opposites in everyday life, and opposites of integer numbers (usually expressed as numbers of equal magnitude, but of opposite direction).

	1	2	3	4	5	6	7	8	9	10
up 5 is opposite of down 5			•	•		•	•	•		
$^{+}5$ is opposite of $^{-}5$	•	•	•	•	•	•	•	•		•
implicit									•	

Most texts covered this notion well.

[10] Cancellation (Neutralization) of Opposites of equal Magnitude

Going up 5, then down 5, returns you to your original position, and therefore has no overall effect.

	1	2	3	4	5	6	7	8	9	10
explicit notion										
implicit notion		•			•					
not at all	•		•	•		•	•	•	•	•

Text 5 did not include this notion until after addition.

[11] Zero as a Neutral Element

Zero is no longer seen as "nothing", but is an element created by combining opposites of equal magnitude. This element has no effect (as the zero in N), but is a "sum" rather than the "nothing" of subtraction.

	1	2	3	4	5	6	7	8	9	10
explicit notion	•			•				•		
implicit notion		•								
not at all			•		•	•	•		•	•

Text 2 calls this the "zero property".

[12] Equivalence Class of Zero

Zero can be represented in many ways, i.e. $^{+}n + ^{-}n = 0$, so any two opposites combine to make zero.

	1	2	3	4	5	6	7	8	9	10
explicit notion								•		
implicit notion		•								
not at all	•		•	•	•	•	•		•	•

This is an important notion that was surprisingly not dealt with by most texts.

[13] Equivalence Class of Any Integer

Any integer may be represented in many forms, using the notion that adding zero to it does not change its value, and since zero exists in many forms, then any integer may be added to any form of zero, and thus be represented in many ways.

	1	2	3	4	5	6	7	8	9	10
explicit notion					•					
implicit notion		•								
not at all	•		•	•		•	•	•	•	•

Text 5 does not present this notion until after addition. This notion is important for one of the models for operations.

OVERALL RATING ON INTEGER NOTIONS: Text 6 (Mathscope 1) is the best text, covering all of the essential notions (omitting criteria 8, 11, 12, 13). The poorest text was number 9 (Scott, Foresman Mathematics).

PART TWO - INTEGER OPERATIONS OF ADDITION & SUBTRACTION AND CORRESPONDING NOTIONS ABOUT THESE OPERATIONS

A: PROCEDURES

[1] Initial Presentation of Integer Addition

The number line for addition is an abstract model which uses the "rule" of a displacement to the right if the number is positive, and to the left if it is negative. To add is to combine two displacements (the first one starting at zero, the second starts where the first ends, and the answer is found by the displacement between zero and the last arrow). The annihilation model is a concrete model which uses chips of different colors for positives and negatives, and physical combining of these two groups. Equal numbers of positives and negatives cancel each other out, and the sum is whatever is left over. A middle position is to refer to everyday life (combining debts and profits), or to set up a pattern, treating positive numbers as natural numbers (such as $3 + 2 = \square$, $3 + 1 = \square$, $3 + 0 = \square$, $3 + {}^-1 = \square$, etc.), and the most abstract approach is to just give rules for addition.

	1	2	3	4	5	6	7	8	9	10
number line	*	1 - *	2 - *	2 - *		*	*		*	1 - *
annihilation		2 - *		1 - *	1 - *			*		
real-life			1 - *		2 - *					2 - *
patterns			3 - *							
rules										

The numbers before the * refer to which model was used first, second, etc. Showing more than one model may be confusing to the student, or it may help to reinforce the notion of addition of integers, rather than causing the notion to be model-specific. It also gives the teacher more choices.

[2] Mathematization of Addition of Integers (leave the model, add numbers)

Models are used to add positive and negative quantities (displacements or chips or profits), but mathematization occurs when integer numbers are able to be added without reference to the concrete model (although a mental model may be used - for example decomposition is a procedure with numbers that parallels the annihilation procedure for addition). Note that not all texts included this step.

	1	2	3	4	5	6	7	8	9	10
rule given then proven										
rule derived from model	•			•			•		•	•
rule given, no reference										
decomposition method								•		
no rules, no mathematization		•	•		•	•				

[3] Initial Presentation of Subtraction

This is the operation that every study has identified as the least successfully performed. Use of the number line here usually depends on trusting that addition and subtraction are inverse operations in this new set, so to subtract a number, the displacement is in the opposite direction that it would be for addition (add +3 is right, so subtract +3 is left; add -5 is left, so subtract -5 is right). The annihilation model uses the "take away" procedure for subtraction. Some texts use the "missing addend" which transforms the subtraction question into an addition one (for example, $^{-}3 - {}^{+}2 = \square$ becomes $^{+}2 + \square = {}^{-}3$).

	1	2	3	4	5	6	7	8	9	10
number line										
annihilation		*			*					
missing addend	1 - *		1 - *			*	*	*		
pattern development	2 - *		2 - *							2 - *
real-life situations				*						
rule given									*	1 - *

Very few texts (only 3) dealt with subtraction as an operation in Z, instead they related it to addition, a relationship not yet proven to exist in Z.

[4] Mathematization of Subtraction of Integers

	1	2	3	4	5	6	7	8	9	10
rule given and proven				*			*	*		
rule derived from model	*		*		*	*				
rule given - no reference										
no rule		*							*	*

Either this step was not dealt with, or reliance on the subtraction rule was encouraged.

B: NOTIONS ABOUT OPERATIONS IN Z

[5] All subtractions are now allowed

	1	2	3	4	5	6	7	8	9	10
explicit notion				*						
implicit notion	*									
not at all		*	*		*	*	*	*	*	*

It is discouraging to see that only one text deals with this notion openly - "subtraction is always possible" - when for 6 years subtraction has been a restricted operation for the student.

[6] Results of Operations

In N, addition has been performed to make larger (or the same - add zero), and subtraction to make smaller. Now in Z, these operations are completely opened up - each operation may make smaller, the same, or bigger.

None of the texts raised this issue.

[7] Equivalence of Addition and Subtraction as Operations

There are now two ways to reach an end result: you may add a certain amount, or you may subtract the opposite (or, stated another way, you may subtract a certain amount, or add the opposite).

	1	2	3	4	5	6	7	8	9	10
only for subtraction			•	•	•	•	•	•	•	•
not at all	•	•								

This notion was usually given or developed by patterns, and only to generate the famous subtraction rule of changing subtraction to addition of the opposite.

OVERALL RATING ON OPERATIONS PRESENTATION: Again, the worst text is text 9, and texts 1 (Holt Mathematics 1), 4 (Mathématique Soleil 1) and 8(Réalités Mathématiques 1) were all equally well rated, although the author's preference goes to text 8 since it uses the annihilation model.

TEXTBOOK EXERCISES numbers refer to number of tasks given)

	1	2	3	4	5	6	7	8	9	10
integers in real situations	•	•	•	•	•	•	•		•	•
identify or write opposites				•	•	•	•	•	•	•
ordering: refer to a model	24			1		13				
ordering: free questions	16	18	70	14	23		18		17	34
addition: refer to a model	8	32	43	12	29	25	8	30	6	13
addition: refer to a rule	50					12		15		
addition: free questions			28	20	24	26	24	36	44	47
addition: word problems	6	3	3	17	3	5	7		1	4
addition: large numbers	38		1	7			3	8	3	
subtraction: refer to model	28	36	30	10	30		10	27		4
subtraction: refer to a rule	50		7	20	6			10	12	6
subtraction: free questions			20	20	6		28		16	24
subtraction: word problems	6			13	2		3		2	
subtraction: large numbers	30			10					3	

OVERALL RATING ON EXERCISE SETS: Based on criteria that operation strategies should end up being free of a model, and that there should be sufficient subtraction practise, the text that has the best number of "free" exercises, and large numbers is text 4 (Mathématique Soleil 1).

List of Textbooks Evaluated:

(1) Holt Mathematics 1, Holt, Rinehart & Winston of Canada, Ltd., Elliot et al., 1976, pp. 152-169, 294-317.

(2) Making Mathematics 7, Gage, Flewelling et al., 1991, pp. 347-411.

(3) Math is 1, Second edition, Nelson, Canada, Ebos, Robinson & Tuck, 1982, pp. 389-409.

(4) Mathématique Soleil 1, Guérin, Montreal, Drolet & Rochette, 1984, pp. 79-103.

(5) Mathquest 7, Addison-Wesley, Brendan Kelly (series editor), 1989, pp. 412-439.

(6) Mathscope 1, Prentice-Hall Canada Inc., H. Laurence Ridge, editor, 1981, pp. 375-394.

(7) McGraw-Hill Mathematics 7, Webster Division, McGraw-Hill Book Co., Max A. Sobel (consulting editor), 1981, pp. 303-326.

(8) Réalités Mathématiques 1, ERPI (Editions du Renouveau Pédagogique), Montreal, Gaudreau & Bourget, 1979, pp. 344-359.

(9) Scott, Foresman Mathematics (grade 7), Scott, Foresman & Co.,.Illinois, 1980, pp. 356-376.

(10) Houghton Mifflin Mathematics 7, Houghton Mifflin Canada, Fran Seidenberg (editor), 1985, pp. 340-349.

APPENDIX C

CLASS ASSESSMENT WORKSHEETS

FIRST CLASS ASSESSMENT

1. Fill in the boxes with the missing numbers:

(a) $14 + 72 + 11 = 72 + 11 + \boxed{}$ *test commutativity for integer subtractions*

(b) $36 + 7 - 13 = 36 + (\boxed{} - 13)$ *test associativity for integer additions*

(c) $(310 + 98) + 54 = 310 + (98 + \boxed{})$ *test associativity for integer additions*

(d) $43 = 13 + \boxed{}$ *test decomposition for integer additions*

(e) $62 = \boxed{} + 19$ *test decomposition for integer additions*

(f) $2 - 8 = \boxed{}$ *test for existence of negative numbers as solutions to subtractions*

(g) $15609 - 15609 = \boxed{}$ *test for subtraction of equal amounts*

(h) $623 - 184 = \boxed{}$ *test for ability to subtract large numbers*

(i) $137 + 98 = \boxed{}$ *test for ability to add large numbers*

2. Put in the sign < or > for these comparisons:

a) 78 ___ 34 b) 14 ___ 24 *test for ability to order, and to use correct symbol.*

3. Fill in the missing number:

$$279 + 814 = 1093$$

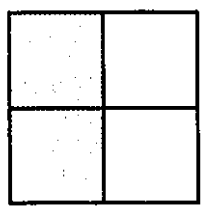
test for reversibility of addition and subtraction

$$1093 - \boxed{} = 279.$$

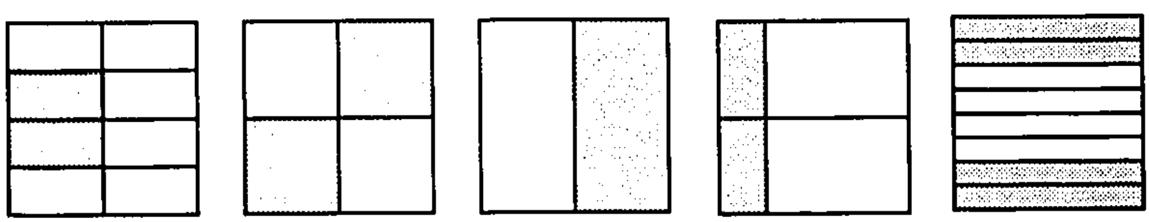
4. Put these numbers on a number line: 4, 0, 10, 7

test ability to draw a number line, to correctly label it, and to correctly plot natural numbers on it.

5. <u>Some</u> of the following diagrams represent the same fraction as



Write YES below the diagrams that show the same fraction.



________ ________ ________ ________ ________

test notion of equivalence classes non-numerically.

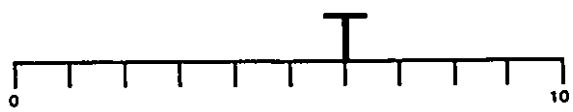
6. Have you ever seen a number like $^{-}5$ before? Sometimes it is written as (-5) or as -5.

(Put an x): Yes ☐ No ☐

If you have, what do you think it means?

test to see whether or not integer notation is familiar to the student, and to see what it means for them if it is.

7. Use a number to tell where the letter T is.



test identification of position with number

8. The thermometer reading on Dec. 1 was 5 degrees. During the night, the temperature fell by 8 degrees. What was the reading on the thermometer at that time?

test notion of an extended number line, and the notion of zero as origin, (and the intuitive use of the concept of negative number in a real-life setting)

9. Joy had a sore throat, so her mother prepared a hot drink of water with lemon for her. Joy tried it and found that it tasted very sour. Her mother added some honey to <u>neutralize</u> the taste.

What does the underlined word mean?

test notion of neutralization as an action. (note that although chemically honey only masks a sour taste, the idea of bringing the taste to one that seems to be neither sweet nor sour is an intuitive notion of neutralization)

10. Premier Bourassa wants to hold a referendum (a vote where everyone chooses either yes or no) to see if the people who live in

Quebec want to stay in Canada, or if they want Quebec to be a separate country. Everyone will get a chance to vote yes or no for separation. The grade 11 classes discussed the issue, since some of them are old enough to vote. Sandra said that she would vote no - she wants Quebec to stay in Canada. Evelyn would vote yes - she thinks Quebec should be separate. Jerry is neutral about the issue.

What does the underlined sentence mean?

test the concept of the state of being neutral as having no effect.

11. Can you think of some opposites? Write them in the space below.

test the notions of a) pairs b) opposite nature

12. Write the opposite of the following:

tall ____________ 5 steps up ____________

10 degrees below zero ____________ succeed ____________

test the notion of equal magnitudes, and opposite nature

SECOND CLASS ASSESSMENT

13. Use the symbol $>$ or $<$ for these comparisons:

(a) -5 ___ -1 (b) 6 ___ -6 (c) -2 ___ 0

test the ability to order integers

14. Write the opposite of the following:

(a) seven kilometers east ____________

confirm notion of opposites (a previous question (12) had used a numeral for the number, and results seemed poorer than anticipated.)

(b) 5 ______

test notion of negative numbers as opposites of positive numbers

15. Fill in the boxes with the missing numbers:

(a) 2 + -7 = $\square$

(b) -6 + 1 = $\square$

(c) -6 - 4 = $\square$

(d) 10 + -2 = $\square$

(e) 4 - -1 = $\square$

test ability to operate on a negative number and a whole number

16. Put these numbers on a number line: -1 7 0 5 -7

test notion of integer number line (some students had spontaneously drawn one during the first pretest)

17. Put the missing numbers.in the boxes:

(a) $485 - 376 + 57 + 376 = \boxed{}$

(b) $286 + 91 - \frac{286}{2} + 79 = \boxed{}$

test notion of cancellation

THIRD CLASS ASSESSMENT

18. In front of you, there is a large container filled with pennies, nickels, dimes and quarters. You must give your teacher 20 cents. What coins will you give him?

____

How many other ways can you find to make 20 cents? List them.

retest notion of equivalence

APPENDIX D

PRE-INSTRUCTION INTERVIEW TASKS

$$392 - 143 + 85 + 143 = \boxed{}$$

$$25 + 814 + 322 - 814 = \boxed{}$$

$$647 + 299 - 299 = \boxed{}$$

$$387 + 798 = 1185$$

$$1185 - \boxed{} = 387$$

rich ______________

negative ________________

20 steps back _______________

700 ________________

4 _______

-32

-10 -8

-45 ___ -920 > or < ?

-8 ___ -4

-6 9 2 -4

$20 - -6 = \boxed{}$

$3 + -10 = \boxed{}$

$-9 + 7 = \boxed{}$

$13 + -1 = \boxed{}$

$-5 - -5 = \boxed{}$

$-4 - 16 = \boxed{}$

$21 + -21 = \boxed{}$

$6 - 15 = \boxed{}$

$6 - 15 = \boxed{}$

$21 + -21 = \boxed{}$

$-4 - 16 = \boxed{}$

$-5 - -5 = \boxed{}$

$13 + -1 = \boxed{}$

$-9 + 7 = \boxed{}$

$3 + -10 = \boxed{}$

$20 - -6 = \boxed{}$

INTERVIEW DESIGN - OBSERVATION SHEETS

Name of Student: ______________________

1. Could you please fill in the blank:

1 (a) 392 - 143 + 85 + 143 = $\boxed{}$

ask Please tell me how you got your answer.

If student cancels, ask : You didn't do all the numbers. Can you tell me why?

If student didn't cancel, go to second example and ask Could you find the missing number here?

1 (b) 25 + 814 + 322 - 814 = $\boxed{}$

If student cancels, ask above question.

If student still does not cancel, ask Can you find a shorter way to get the answer?

If still not noticed, do third example

1. (c) 647 + 299 - 299 = $\boxed{}$

2. What is the missing number?

387 + 798 = 1185
1185 - $\boxed{}$ =387

If student makes association and just writes number, ask: How do you know that is the right answer?

How did you get your answer?

If student does the arithmetic, ask: Can you find an easier way to get the answer?

3. What is the opposite of:

(a) rich ____________ why?

(b) negative ____________ why?

(c) 20 steps back ________________ why? Could I put 8 steps forward?

Do you think that **numbers** can have opposites?

(d) Does 700 have an opposite? How is your number opposite?
If student doesn't think 700 has an opposite,
(e) ask Does 4 have an opposite?

4. What do you think this means? (-32) (Write the number, don't say it.) If student does not have a meaning for it, ask What do you think this means? (-9) (again, write the number)

How do you know it means that? Did someone tell you?

If they generalize (it's lower than zero, it's a negative number, it's an answer to subtraction, etc.) ask: Can you give me an example.(or illustration) i.e. Can you show me what you mean?

5. (a) One of your classmates asks you: "Which of these two numbers is larger?" (-10, -8) What would you tell him?

How would you convince him if he didn't believe your answer?

(b) Use one of these signs (> or <) in the blank:

-45 ____ -920

If unable, or unsure, use -8 ___ -4 with the same question.
Can you read the answer for me? How did you decide which sign to use?

6. I want to put these numbers on a number line: -6, 9, 2, -4.

Tell me how to make a number line, and where to put the numbers.

How do you know where to put the labels on the line?

7. Fill in the blanks: Can you do your thinking aloud so that I can understand how you get your answers?

Note here that since Steven and Robert had already correctly solved most of these operations on the pretest, they were given the questions in the reverse order from the others to see if they were indeed strong, or if they had devised a method of solving from performing the previous tasks from easy to difficult. The others were given the tasks from easy to difficult to see where they would break down, to see how far they could go.

Marilyn, Jason, Richard, Connie	Steven, Robert
(a) $6 - 15 = \square$	(h) $20 - -6 = \square$
(b) $21 + -21 = \square$	(g) $3 + -10 = \square$
(c) $-4 - 16 = \square$	(f) $-9 + 7 = \square$
(d) $-5 - -5 = \square$	(e) $13 + -1 = \square$
(e) $13 + -1 = \square$	(d) $-5 - -5 = \square$
(f) $-9 + 7 = \square$	(c) $-4 - 16 = \square$
(g) $3 + -10 = \square$	(b) $21 + -21 = \square$
(h) $20 - -6 = \square$	(a) $6 - 15 = \square$

APPENDIX E

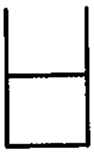
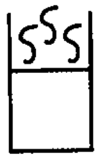
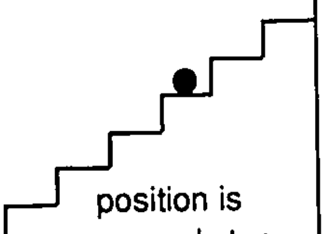
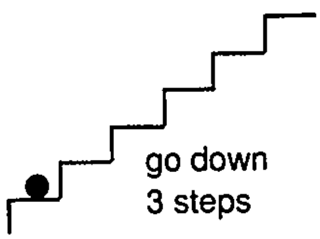
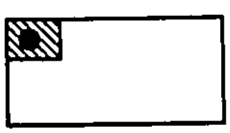
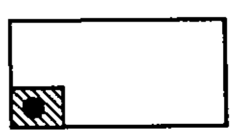
LESSON ONE WORKSHEETS

LESSON ONE - MARCH 9, 1992

(1) Oppositeness

sick	well
dead	alive
backwards	forwards
add	subtract

tall	average	short
weak	average	strong
hot	comfortable	cold
rich	getting by	poor
	neutral position	

NEUTRALITY	CHANGE (TRANSFORMATION)	NEUTRALIZATION (RETURN TO NEUTRALITY)
position is Montreal	fly north 100 km	
 water at room temperature	 add 10 L hot water	
350 amount is 350	subtract 25 350 - 25 = 325	
 balanced seesaw	 add 60 kg to left side	
+ -	add 3 negatives + - - - -	
 position is second step	 go down 3 steps	
	flip downwards 	

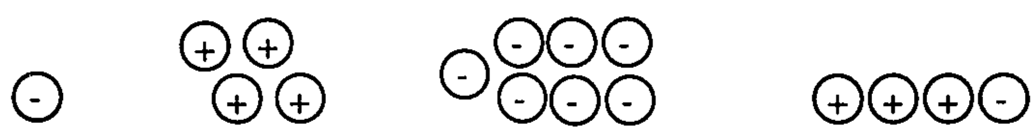
$$21 + 13 - 7 + 9 + 8 - 13 - 8 + 7 - 21 = \square$$

$$18 - 4 + 9 - 12 + 12 - 9 + 17 - 18 + 4 = \square$$

$$27 + 22 - 4 + 18 - 22 - 4 - 7 - 18 + 7 = \square$$

$$617 - 298 + 321 - 617 + 298 = \square$$

WHAT IS THE VALUE OF:



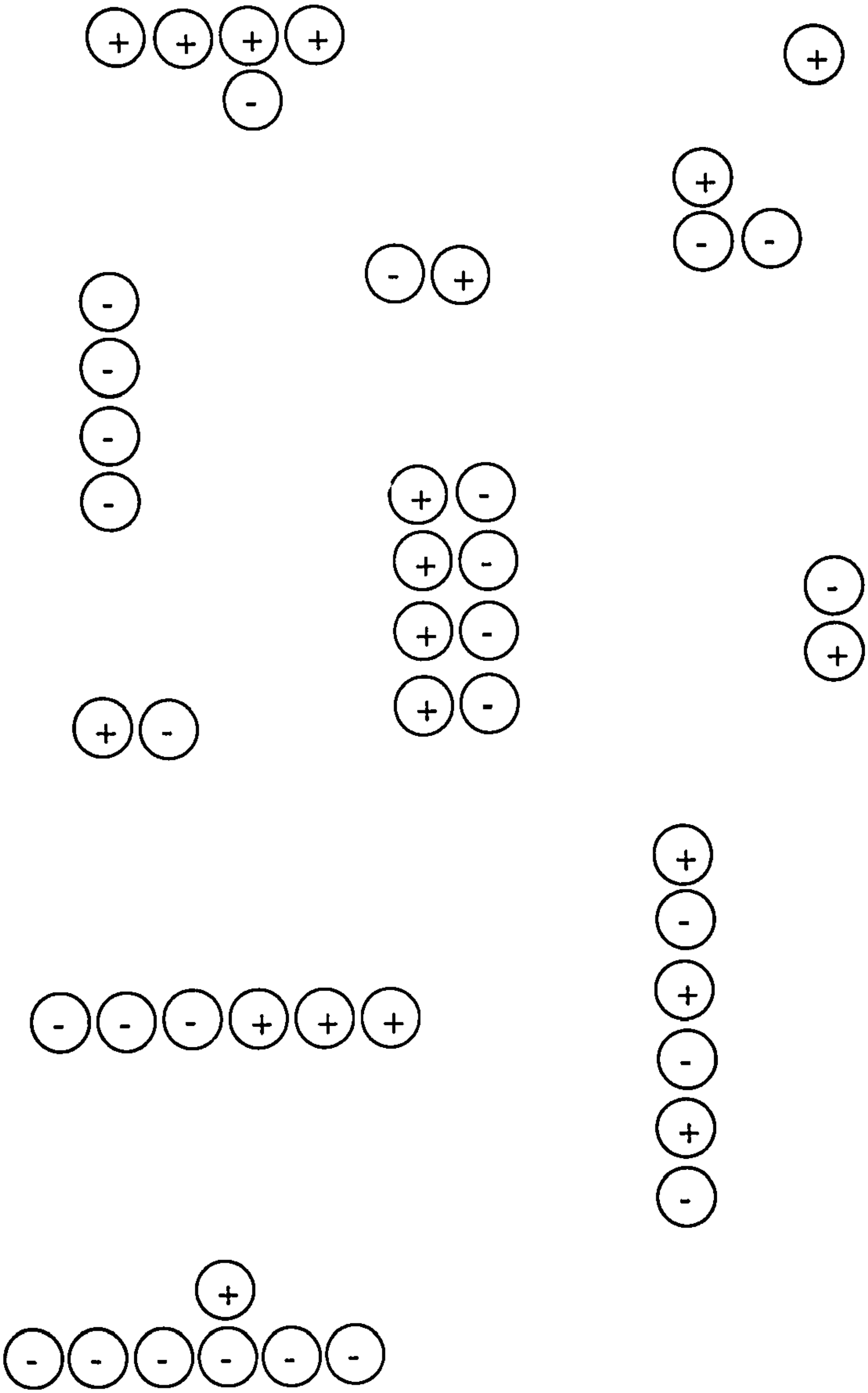
______ ______ ______ ______

ILLUSTRATE THESE VALUES USING + OR - SIGNS:

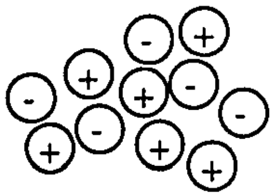
(a) +6 (b) -4

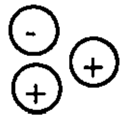
WHAT IS THE MEANING OF:

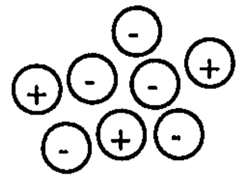
(a) $^{-}713$ (b) $^{+}326$



WHAT IS THE VALUE OF:







DRAW EQUIVALENT REPRESENTATIONS FOR:

- 4	+ 1

APPENDIX F

LESSON TWO WORKSHEETS

LESSON TWO WORKSHEETS

$185 = 130 + \boxed{}$

$215 = \boxed{} + 200$

$87 - 14 + 32 + 14 - 29 - 87 + 29 = \boxed{}$

Write the value of:

a) - - - - - ________

b) + + - + + - - - + + + + - ______

c) a group of chips that has 89 positives and 17 negatives. ____________

$^{-}8 + {}^{-}12 =$

$^{+}6 + {}^{+}5 =$

$^{+}100 + {}^{+}165 =$

$^{-}215 + {}^{-}32 =$

$${}^{-}825 + \boxed{} = 0$$

$$\boxed{} + {}^{-}29 = 0$$

$${}^{+}3145 + {}^{-}3145 = \boxed{}$$

$${}^{-}25 + 0 = \boxed{}$$

$${}^{-}49 + {}^{+}49 + {}^{+}32 = \boxed{}$$

$${}^{-}86 + {}^{+}21 + {}^{-}21 = \boxed{}$$

$^{+}5 + ^{-}2 =$

$^{-}4 + ^{+}18 =$

$^{+}9 + ^{-}15 =$

$^{-}11 + ^{+}6 =$

$^{-}66 + ^{+}50 =$

$^{+}54 + ^{-}100 =$

$^{+}215 + ^{-}5 =$

$^{-}111 + ^{+}130 =$

$^{+}5 + {}^{+}11 =$

$^{-}6 + {}^{+}6 =$

$^{-}19 + {}^{+}4 =$

$^{-}3 + {}^{-}8 =$

$^{-}3 + {}^{-}6 + {}^{-}4 =$

$^{+}18 + {}^{+}9 + {}^{-}9 =$

$^{-}9 + {}^{+}2 + {}^{-}6 =$

APPENDIX G

LESSON THREE WORKSHEETS

Show 4 ways to represent ${}^{+}5$

$${}^{+}18 + {}^{-}43 + {}^{-}18 + {}^{+}13 = \underline{\qquad}$$

$${}^{+}21 + {}^{-}81 = \underline{\qquad}$$

$${}^{-}48 + {}^{-}16 = \underline{\qquad}$$

$${}^{-}49 + \boxed{} = 0$$

$${}^{+}3 + {}^{-}8 = \underline{\qquad}$$

$${}^{-}10 - {}^{-}3 = \underline{\qquad}$$

$^{-}9 \; - \; ^{-}5 \; = \; \_\_\_\_\_\_$

$^{+}21 \; - \; ^{+}11 \; = \; \_\_\_\_\_\_$

$^{+}972 \; - \; ^{+}222 \; = \; \_\_\_\_\_\_$

$^{-}82 \; - \; ^{-}14 \; = \; \_\_\_\_\_\_$

$^{+}67 \; - \; ^{+}67 \; = \; \_\_\_\_\_\_$

$^{-}812 \; - \; ^{-}812 \; = \; \_\_\_\_\_\_$

$^{+}6 \; + \; ^{-}9 \; + \; ^{+}9 \; - \; ^{-}9 \; = \; \_\_\_\_\_\_$

$^{+}3 - {}^{+}7 =$ ______

$^{-}8 - {}^{-}10 =$ ______

$^{-}1 - {}^{-}5 =$ ______

$^{+}3 - {}^{+}11 =$ ______

$^{-}9 - {}^{+}2 =$ ______

$0 - {}^{+}9 =$ ______

$^{+}4 - ^{-}6 =$ ______

$^{-}2 - ^{+}8 =$ ______

$^{+}12 - ^{-}3 =$ ______

$^{+}8 - ^{-}8 =$ ______

$^{-}7 - ^{+}5 =$ ______

$0 - ^{-}7 =$ ______

$^{+}100 - ^{+}145 =$ ______

$^{+}75 - ^{-}20 =$ ______

$^{-}43 - ^{+}68 =$ ______

$^{-}99 - ^{-}153 =$ ______

$^{-}765 - ^{-}219 =$ ______

$^{-}810 - ^{+}810 =$ ______

$^{-}230 - ^{+}105 =$ ______

APPENDIX H

LESSON FOUR WORKSHEETS

$^{-}18 - ^{-}58 =$ ________

$^{-}80 - ^{+}75 =$ ________

$^{+}30 - ^{-}30 =$ ________

$^{+}89 - ^{-}20 =$ ________

$0 - ^{-}45 =$ ________

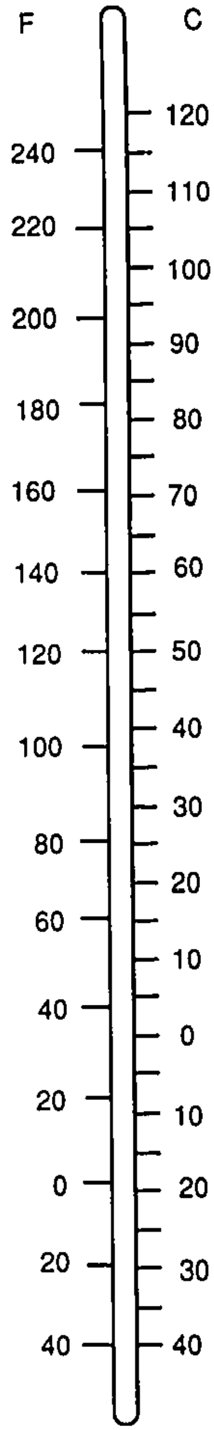
$^{+}30 - {}^{-}95 = $ __________

$^{-}81 - {}^{-}81 = $ __________

$^{-}13 - {}^{+}49 = $ __________

$^{+}42 - {}^{+}100 = $ __________

$^{-}27 - {}^{+}27 = $ __________


F
C
120
240
110
220
100
200
90
180
80
160
70
140
60
120
50
100
40
30
80
20
60
10
40
0
20
10
0
20
20
30
40
40

APPENDIX I

LESSON FIVE WORKSHEETS

$^{+}66 + {}^{-}21 =$ ________

$^{-}85 - {}^{+}34 =$ ________

$^{-}95 - {}^{-}18 =$ ________

$^{+}78 - {}^{+}120 =$ ________

$^{-}99 + {}^{-}53 =$ ________

$119 + 286 = \boxed{}$

$345 + \boxed{} = 800$

$217 - 49 = \boxed{}$

$582 - \boxed{} = 106$

$+45 = +16 + \boxed{}$

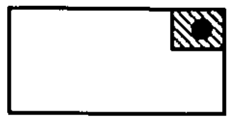
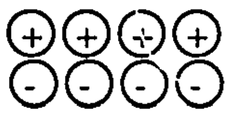
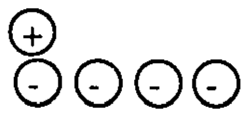
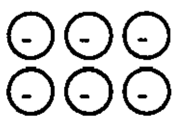
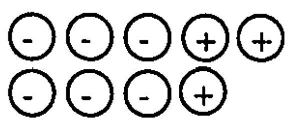
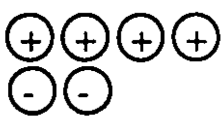
$^{-}82 = \boxed{} + {}^{-}20$

$+6 + \boxed{} = +4$

$-12 - \boxed{} = -3$

$+5 - \boxed{} = {}^{-}2$

$^{-}8 - \boxed{} = +7$

Neutral State	Change	Neutralization
	 add 60 kg to left	____________
	 flip right	____________
	 subtract 3 pos.	____________
	 add 3 positives	____________
	 subtract 2 negs.	____________

Start With	Operation	Neutralization
310	add 99 310 + 99 = 409	
$^{+}12$	add $^{+}5$ $^{+}12 + {}^{+}5 = {}^{+}17$	
$^{-}4$	add $^{-}10$ $^{-}4 + {}^{-}10 = {}^{-}14$	
$^{+}11$	subtract $^{-}3$ $^{+}11 - {}^{-}3 = {}^{+}14$	
$^{-}9$	subtract $^{-}1$ $^{-}9 - {}^{-}1 = {}^{-}8$	
600	subtract 150 600 - 150 = 450	

$^{+}7$	subtract $^{+}10$ $^{+}7 - {}^{+}10 = {}^{-}3$	
$^{-}1$	subtract $^{+}5$ $^{-}1 - {}^{+}5 = {}^{-}6$	
$^{-}6$	add $^{+}1$ $^{-}6 + {}^{+}1 = {}^{-}5$	
$^{+}9$	add $^{-}7$ $^{+}9 + {}^{-}7 = {}^{+}2$	

-10 + +2 -6 + -2 +3 + +1

-10 - -2 -6 - +2 +3 - -1

When I add two numbers, my answer is
___ (a) always bigger than both numbers
___ (b) sometimes bigger than both numbers
___ (c) never bigger than both numbers
(always smaller than both numbers)

When I subtract two numbers, my answer is
___ (a) always smaller than the first number
___ (b) sometimes smaller than the first
number
___ (c) never smaller than the first number
(always bigger than the first number)

APPENDIX J

POST-TEST TASKS

POST-TEST - LESSON SIX

(1) If I gave you any two integers and asked you which one is larger, how would you decide? *test ordering of integers: abstract*

(2) Put these numbers in order: $^{+}7$, $^{-}8$, 0, $^{-}20$, $^{-}2$, $^{+}3$. How did you make your decision? *test ordering of integers: concrete*

(3) Why do we not put a raised plus or minus sign before the zero?
test notion of zero as neutral

(4) What is the opposite of $^{-}216$? ______ why?
test notion of integer opposites

(5) If I gave you any two integers, and asked you to add them, what would you do first? How would you get your answer?
Give me an example with numbers in it.
(If only one case covered) Can you think of an addition that doesn't work that way? How <u>does</u> it work? Give me an example.
test abstraction rules of integer addition

(6) If I gave you any two integers, and asked you to subtract them, what would you do first? How would you get your answer?
Give me an example with numbers in it.
(If only one case covered) Can you think of a subtraction that doesn't work that way? How <u>does</u> it work? Give me an example.
test abstraction rules of integer subtraction

(7) Find the missing numbers:

$^{+}95 = \square + {}^{+}37$

test integer decomposition

$^{-}64 = {}^{-}25 + \square$

$^{+}61 + {}^{-}20 + {}^{-}45 + {}^{-}12 - {}^{-}20 + {}^{+}45 - {}^{+}12 =$ ______ *integer cancellation*

$926 - 78 + 300 + 54 - 82 - 54 - 82 - 926 + 78 =$ _____
whole number cancellation

(8) Answer these questions - tell me how you solve them. (If the large numbers are too difficult, substitute smaller numbers for them - if they request the chips, give them.)
mixed integer addition and subtraction, concrete examples with large numbers

$0 - {}^{-}300 =$ ______

$^{+}219 + {}^{-}19 =$ ________

$^{-}906 - {}^{+}906 =$ ________

$^{+}78 - {}^{+}100 =$ ________

$^{-}218 + {}^{-}51 =$ _______

$^{-}280 - {}^{-}280 =$ ________

$^{-}86 - {}^{+}21 =$ ________

$^{-}817 + {}^{+}817 =$ ________

$^{-}390 - {}^{-}105 =$ ________

$^{+}62 - {}^{-}50 =$ ________

$0 + {}^{-}95 =$ ________

$^{-}820 + {}^{+}1000 =$ ________

$^{+}342 - {}^{-}342 =$ ________

(9) Show how to solve with chips:

$^{-}8 + {}^{+}13$ $\qquad\qquad$ $^{+}6 - {}^{-}4$

integer addition and subtraction, concrete examples with small numbers

(10) Use any number of + and - signs to represent -2. Show another way. Why are they equivalent?

test equivalence of integer representations

(11) I started with $^{+}3$. I ended up with $^{-}2$. What did I do? Write the sentence. Is there any other way I could have done it? Write the sentence.

I started with $^{-}9$. I ended up with $^{-}5$. What operation did I make? Write the sentence. Is there any other way I could have done it? Write the sentence.
(If they only get one way for the first, and 2 for the second, go back to the first and ask if they're sure there's only one way.)

test awareness of results of integer operations

$^{+}7, \quad ^{-}8, \quad 0, \quad ^{-}20, \quad ^{-}2, \quad ^{+}3$

$^{+}95 = \boxed{} + {}^{+}37$

$^{-}64 = {}^{-}25 + \boxed{}$

$^{+}61 + {}^{-}20 + {}^{-}45 + {}^{-}12 - {}^{-}20 + {}^{+}45 - {}^{+}12 =$ ___

$926 - 78 + 300 + 54 - 82 - 54 - 82 - 926 + 78 =$ _______

$0 - {}^{-}300 = \_\_\_\_\_\_$

$^{+}219 + {}^{-}19 = \_\_\_\_\_\_$

$^{-}906 - {}^{+}906 = \_\_\_\_\_\_$

$^{+}78 - {}^{+}100 = \_\_\_\_\_\_$

$^{-}218 + {}^{-}51 = \_\_\_\_\_\_$

$^{-}280 - {}^{-}280 = \_\_\_\_\_\_$

$^{-}86 - {}^{+}21 =$ ________

$^{-}817 + {}^{+}817 =$ ________

$^{-}390 - {}^{-}105 =$ ________

$^{+}62 - {}^{-}50 =$ ________

$0 + {}^{-}95 =$ ________

$^{-}820 + {}^{+}1000 =$ ________

$^{+}342 - {}^{-}342 =$ ________

$^{-}8 + {}^{+}13$

$^{+}6 - {}^{-}4$

$^{-}2$